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NEW FORMULAS FOR DECREASING REARRANGEMENTS AND A CLASS OF ORLICZ-LORENTZ SPACES

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Abstract. Using a nonlinear version of the well known Hardy-Littlewood inequalities, we derive new formulas for decreasing rearrangements of functions and sequences in the context of convex functions. We use these formulas for deducing several properties of the modular functionals defining the function and sequence spaces $M_{\varphi,w}$ and $m_{\varphi,w}$ respectively, introduced earlier in [3] for describing the Köthe dual of ordinary Orlicz-Lorentz spaces in a large variety of cases (φ is an Orlicz function and w a *decreasing* weight). We study these $M_{\varphi,w}$ classes in the most general setting, where they may even not be linear, and identify their Köthe duals with ordinary (Banach) Orlicz-Lorentz spaces. We introduce a new class of rearrangement invariant Banach spaces $\mathcal{M}_{\varphi,w}$ which proves to be the Köthe biduals of the $M_{\varphi,w}$ classes. In the case when the class $M_{\varphi,w}$ is a separable quasi-Banach space, $\mathcal{M}_{\varphi,w}$ is its Banach envelope.

1. Introduction and Preliminaries

The theories of Lorentz, Orlicz-Lorentz or Marcinkiewicz spaces have been developed on the ground of the concept of decreasing rearrangement, submajorization and maximal functions. A basic tool in this domain are the Hardy-Littlewood inequalities and equations which are thoroughly discussed in several papers and monographs [1, 12]. In this paper we study special Orlicz-Lorentz classes which were introduced for the purpose of expliciting the structure of the Köthe duals of a large variety of "classical" Orlicz-Lorentz spaces [3]. It turns out that in this context a certain kind of "generalized Hardy-Littlewood inequalities", in the spirit of those introduced in the fifties by G. G. Lorentz [14], are particularly useful.

Let's first introduce basic notions, definitions, symbols and facts needed later. As usual by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ we denote the set of all real, non-negative real and natural numbers, respectively. Let $I = (0, a), 0 < a \le \infty$. By L^0 denote the set of all Lebesgue measurable functions $f : I \to \mathbb{R}$. Given $f \in L^0$ define its *decreasing rearrangement* as

$$f^*(t) = \inf |\{s \in I : d_f(s) \le t\}|, \quad t \in I,$$

where $d_f(s) = |\{t \in I : |f(t)| > s\}|$, $s \ge 0$. Analogously, if $x = \{x(n)\}$ is a sequence of real numbers then $x^* = \{x^*(n)\}$, where

$$x^*(n) = \inf\{s > 0 : d_x(s) \le n - 1\}, \quad n \in \mathbb{N},$$

and $d_x(s) = |\{k \in \mathbb{N} : |x(k)| > s\}|$, $s \ge 0$. Given two functions $f, g \in L^0$, or respectively two sequences x, y, we write $f \sim g$, respectively $x \sim y$, whenever $f^* = g^*$, respectively $x^* = y^*$.

Given $f, g \in L^0$, we say that g is *submajorized* by f and write $g \prec f$, whenever

$$\int_0^t g^* \le \int_0^t f^* \quad \text{for all} \quad t \in I.$$

Similarly for sequences $x = \{x(n)\}$, $y = \{y(n)\}$ we write $y \prec x$, if

$$\sum_{n=1}^m y^*(n) \le \sum_{n=1}^m x^*(n) \quad \text{for all} \quad m \in \mathbb{N}.$$

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By $|A|$ we shall denote the cardinality of $A \subset \mathbb{N}$ or the Lebesgue measure of A for $A \subset I$. Recall that a function $\tau : I \to I$ is a *measure preserving transformation* [1] if for every measurable set $A \subset I$ the set $\tau^{-1}(A) = \{t \in (0,a) : \tau(t) \in A\}$ is measurable and $|\tau^{-1}(A)| = |A|$. It is well known that for any two measurable sets $A, B \subset I$ with $|A| = |B|$ there is a measure preserving transformation $\tau : A \to B$ which is measurable, one-to-one and onto function [18]. Any one-to-one and onto function of $\mathbb{N}$ will be called *automorphism* of $\mathbb{N}$, and obviously it preserves the measure of $A \subset \mathbb{N}$.

The space $(E, \|\cdot\|_E)$ is called a (quasi) Banach function space if $E \subset L^0$, $\|\cdot\|_E$ is a (quasi) norm and whenever $f \in E$, $g \in L^0$ and $|g| \le |f|$ then $g \in E$ and $\|g\|_E \le \|f\|_E$. We say that a Banach function space E is a rearrangement invariant (r.i.) function space whenever $f \in E$ yields that $f^* \in E$ and $\|f\|_E = \|f^*\|_E$. We say that $(E, \|\cdot\|_E)$ satisfies the *Fatou property* if for any $f_n \in E$ if $f_n \uparrow f$ a.e. and $\sup_n \|f_n\|_E < \infty$ then $f \in E$ and $\|f_n\|_E \uparrow \|f\|_E$. The space E is said to be *order continuous* if $\|f_n\|_E \downarrow 0$ for any $f_n \in E$ such that $f_n \downarrow 0$ a.e. Analogously $(E, \|\cdot\|_E)$ is called a (quasi) Banach sequence space or rearrangement invariant sequence space if E is a subspace of all real sequences and has the similar properties as analogous function spaces. For information on Banach function or sequence spaces we refer to [1, 10, 12, 13, 19] and for quasi-Banach spaces to [8, 4, 9].

The terms decreasing or increasing will stand for non-increasing or non-decreasing, respectively. A function $w \in L^0$ is called a *weight function* whenever it is non-negative and decreasing. We set $W(t) = \int_0^t w$ for all $t \in I$. The function W is either everywhere infinite (except at 0) or everywhere finite on I. Similarly $w = \{w(n)\}$ is a *weight sequence* if it is non-negative and decreasing. Let also $W(n) = \sum_{i=1}^n w(i)$, $n \in \mathbb{N}$. Notice that the function $W(t)/t$ on I, and the sequence $\{W(n)/n\}$ are decreasing. It is said that the weight function or the weight sequence w is *regular* if there exists $C > 0$ such that $W(u) \le Cuw(u)$, for all $u \in I$ or $u \in \mathbb{N}$, respectively.

That weight functions and sequences are assumed to be decreasing will be recalled only in the statements of important results.

The symbol φ throughout the paper stands for an *Orlicz function* that is $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, strictly increasing and $\varphi(0) = 0$. It follows that $\varphi(t)/t$ is increasing and $\varphi(c/t)t$ is decreasing with respect to $t > 0$, for any $c > 0$. It is said that φ satisfies *condition* Δ_2 whenever there exists $K > 0$ such that $\varphi(2u) \le K\varphi(u)$ for all $u \ge 0$. Let $\varphi_*(t) = \sup_{s>0}\{st - \varphi(s)\}$, $t \ge 0$, be the *complementary* function of φ. An Orlicz function φ is called *N-function* whenever $\lim_{t\to\infty} \varphi(t)/t = \infty$ and $\lim_{t\to 0} \varphi(t)/t = 0$. It is well known that the conjugate φ_* of an N-function φ is also an N-function [11]. We say that two Orlicz functions φ_1 and φ_2 are equivalent if for some $C > 0$, $\varphi_1(C^{-1}t) \le \varphi_2(t) \le \varphi_1(Ct)$ for all $t \ge 0$.

We also say that two expressions A and B (not both Orlicz functions) are equivalent, whenever there exists $C > 0$ such that $C^{-1}A \le B \le CA$.

In the context of the work presented here let's agree on the following convention. Given $f, v \ge 0$ on I, if $v(t) = 0$ then define

$$\varphi\left(\frac{|f(t)|}{v(t)}\right) v(t) = \begin{cases} 0 & \text{if } f(t) = 0; \\ \infty & \text{if } f(t) \ne 0. \end{cases}$$

Similarly as above, for sequences $v = \{v(n)\} \ge 0$ and $x = \{x(n)\}$, if $v(n) = 0$ define

$$\varphi\left(\frac{|x(n)|}{v(n)}\right) v(n) = \begin{cases} 0 & \text{if } x(n) = 0; \\ \infty & \text{if } x(n) \ne 0. \end{cases}$$

For any function $f \in L^0$ and sequence $\{x(n)\}$ let

$$M(f) = \int_I \varphi\left(\frac{f^*}{w}\right) w \quad \text{and} \quad m(x) = \sum_{n=1}^{\infty} \varphi\left(\frac{x^*(n)}{w(n)}\right) w(n).$$

Let's now discuss the results of this paper. In section 2 we discuss new rearrangement invariant formulas, expressing $M(f)$ or $m(x)$ in an equivalent way, in the spirit of Hardy-Littlewood formulas [12, pp. 63-75], [1, Chapter 2, sec. 2-3]. Recall that the basic Hardy-Littlewood formulas state that for any $f, g \in L^0$, $\int_I fg \leq \int_I f^* g^*$, and in fact we have that $\int_I f^* g^* = \sup_{g\sim v} \int_I |f||v|$. There exists also a parallel formula if one of the function is increasing and the other is decreasing. So if $h \geq 0$ is increasing then $\int_I f^* h = \inf_{g\sim h} \int_I |f||g|$. On the other hand G. G. Lorentz [14] extended the Hardy-Littlewood inequalities as $\int_I \Phi(f,g) \leq \int_I \Phi(f^*, g^*)$, where the interval I is finite and $\Phi : \mathbb{R}_+^2 \to \mathbb{R}$ belongs to a certain class of continuous functions which are called nowadays "supermodular". This means that, for the natural lattice structure of $\mathbb{R}^2$:

$$\forall x, y \in \mathbb{R}_+^2, \Phi(x \vee y) + \Phi(x \wedge y) \geq \Phi(x) + \Phi(y)$$

These "generalized Hardy-Littlewood inequalities" were extended in the last decade to n-dimensional context and to other kinds of rearrangement procedures, with applications to the theory of optimization and partial differential systems.

We show first that given an Orlicz function φ and a weight function w, $M(f) \leq \int \varphi(|f|/v)v$ for any $v \sim w$, $v \geq 0$, and then that $M(f) = \inf_{v\sim w} \int \varphi(|f|/v)v$. The work is conducted initially for finite sequences, then extended to infinite sequences and eventually to function case. The link with the above cited result by G.G. Lorentz is the fact that the function $(s,t) \mapsto -\varphi(\frac{s}{t})t$ is supermodular on the interior of $\mathbb{R}_+^2$. However it takes value ∞ on the semi-axis $\{t = 0\}$. For this reason and for the commodity of the reader, our proof is self-contained and does not refer to [14]. Next, under additional assumption of w being "controlled by φ", in particular under regularity of w and Δ_2 condition of φ, we refine the approximation of $M(f)$ by similar infimum expression limited to positive v and such that $v \sim w$. We close the section showing another formula for $M(f)$ that states $M(f) = \inf_{v^* \leq w} \int \varphi(|f|/v)v$. Analogous formulas are also proved for $m(x)$.

By $M_{\varphi,w}$ and $m_{\varphi,w}$ denote the class of all functions $f \in L^0$ and sequences $x = \{x(n)\}$, such that for some $\lambda > 0$, $M(\lambda f) < \infty$ and $m(\lambda x) < \infty$, respectively. Given $f \in M_{\varphi,w}$ and $x \in m_{\varphi,w}$, define

$$\|f\|_M = \inf\{\epsilon > 0 : M(f/\epsilon) \leq 1\} \quad \text{and} \quad \|x\|_m = \inf\{\epsilon > 0 : m(x/\epsilon) \leq 1\}.$$

Note that these classes are never trivial. They contain indicator functions of integrable sets, and more generally bounded measurable functions with supports of finite measure. Indeed the function $\varphi(1/w)w$ is increasing on I, and thus $\int_0^t \varphi(1/w)w < \infty$ for every $t \in I$. The notations $\|\cdot\|_M$ and $\|\cdot\|_m$ should not be misleading since these functionals are in general neither norms nor equivalent to norms. They are not even necessarily equivalent to quasi-norms, and the classes $M_{\varphi,w}$ and $m_{\varphi,w}$ may not be even linear spaces. Nevertheless the set of positive decreasing elements in these classes is a convex cone, on which the functionals $\|\cdot\|_M$ and $\|\cdot\|_m$ are convex. Section 3 is devoted to investigate these classes.

The classes $M_{\varphi,w}$ and $m_{\varphi,w}$ appeared for the first time in [3] as duals of Orlicz-Lorentz spaces. It was proved that under assumptions of regularity of w, if φ is an N-function then the Köthe dual $(\Lambda_{\varphi,w})'$ of Orlicz-Lorentz space $\Lambda_{\varphi,w}$ coincides with $M_{\varphi_*,w}$ as sets and the homogeneous functional $\|\cdot\|_{M_{\varphi_*,w}}$ is equivalent to the dual norm in $(\Lambda_{\varphi,w})'$. In this section we study the more general case where w is no more assumed to be regular, nor integrable on finite intervals. In this wider context it may happen that the functionals $\|\cdot\|_M$ and $\|\cdot\|_m$ are not quasinorms. We show that in this case the classes $M_{\varphi,w}$, $m_{\varphi,w}$ are not even closed under addition. This phenomenon was first described for the case of generalized Lorentz spaces in the paper [2]. A sufficient condition for these classes to be quasi-normed Banach spaces is that $1/w$ satisfies condition Δ_2. This condition is also necessary when φ has lower index $\alpha_\varphi > 1$. But even when this is not the case, it happens that these classes have nontrivial Köthe duals provided w is integrable on finite intervals, and these duals are precisely the ordinary Orlicz-Lorentz spaces $\Lambda_{\varphi_*,w}$ corresponding to the conjugate Orlicz function φ_*. This may be considered as a kind of generalization of the main result of [3].

In section 4, inspired by the formulas proved in section 2, for any Orlicz function φ and a weight w we introduce a new class of function spaces $\mathcal{M}_{\varphi,w}$ and their sequence version the spaces $\mathfrak{m}_{\varphi,w}$. We first show that they are rearrangement invariant Banach spaces with the Fatou property. It holds in general that $M_{\varphi,w} \subset \mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w} \subset \mathcal{M}_{\varphi,w}$. We prove that they are equal with the quasinorm $\|\cdot\|_M$ (resp., $\|\cdot\|_m$) and the norm in $\mathcal{M}_{\varphi,w}$ (resp., in $\mathfrak{m}_{\varphi,w}$) equivalent whenever w is regular. The converse of this statement holds true under additional assumption that the lower index $\alpha_\varphi > 1$. The latter fact was obtained by calculating the formula of the fundamental function of the spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$. We finish this section by showing that the spaces $\mathcal{M}_{\varphi,w}$ are the Köthe duals of the ordinary Orlicz-Lorentz spaces $\Lambda_{\varphi_*,w}$, and thus the Köthe biduals of the spaces $M_{\varphi,w}$ where this last identification is isometric. It follows at once that if $M_{\varphi,w}$ is normable then $M_{\varphi,w} = \mathcal{M}_{\varphi,w}$ with equivalent norms.

The identification of $\mathcal{M}_{\varphi,w}$ as Köthe dual of $\Lambda_{\varphi_*,w}$ was first given by K. Leśnik, after attending a talk about a first version of the present paper. Leśnik's elegant proof is based on Calderón-Lozanovskii method and quite different from that given here. It will be presented in the paper [5] where the theory of these new classes $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$ is developed further.

2. Formulas for Decreasing Rearrangements

We define $[[1,n]] = \{1,\dots,n\}$ for any $n \in \mathbb{N}$.

Proposition 2.1. *Let $w = \{w(i)\}_{i=1}^n$ be a finite decreasing, positive weight sequence. Then for every finite sequence $x = \{x(i)\}_{i=1}^n$,*

$$\sum_{i=1}^n \varphi\left(\frac{x^*(i)}{w(i)}\right) w(i) = \inf\left\{\sum_{i=1}^n \varphi\left(\frac{x^*(i)}{w(\sigma(i))}\right) w(\sigma(i)) : \quad \sigma \textit{ is a permutation of } [[1,n]]\right\}$$

Proof. Clearly, the left side of the required inequality is bigger than the right side. It remains to show the opposite inequality. Let's start with two dimensional spaces, and take $x^* = (s_1, s_2)$ and $w = (t_1, t_2)$, where $s_1 > s_2 > 0$ and $t_1 > t_2 > 0$. Then we have $s_1/t_1 > s_2/t_1$, $s_1/t_2 > s_2/t_2$ and $s_1/t_2 > s_1/t_1$, $s_2/t_2 > s_2/t_1$. Consequently by convexity of φ we get

$$\frac{\varphi(s_1/t_1) - \varphi(s_2/t_1)}{s_1/t_1 - s_2/t_1} \le \frac{\varphi(s_1/t_2) - \varphi(s_2/t_2)}{s_1/t_2 - s_2/t_2},$$

which equivalently means that

$$\sum_{i=1}^2 \varphi\left(\frac{x^*(i)}{w(i)}\right) w(i) \le \sum_{i=1}^2 \varphi\left(\frac{x^*(i)}{w(\sigma(i))}\right) w(\sigma(i)), \tag{1}$$

where $\sigma(1) = 2$ and $\sigma(2) = 1$. Now we reason by induction on n. We assume for $n > 2$ that

$$\sum_{i=1}^{n-1} \varphi\left(\frac{x^*(i)}{w(i)}\right) w(i) \le \sum_{i=1}^{n-1} \varphi\left(\frac{x^*(i)}{w(\alpha(i))}\right) w(\alpha(i))$$

for any permutation α of $[[1,n-1]]$. If $\sigma(n) = n$, then σ induces a permutation of $[[1,n-1]]$ and we apply the induction hypothesis to $\{x(i)\}_{i=1}^{n-1}$ and $\{w(i)\}_{i=1}^{n-1}$. Adding to both sides the term $\varphi\left(\frac{x^*(n)}{w(n)}\right) w(n)$ we get the required inequality. If $\sigma(n) < n$, then also $\sigma^{-1}(n) < n$. By (1) setting $s_1 = x^*(\sigma^{-1}(n))$, $s_2 = x^*(n)$, $t_1 = w(\sigma(n))$, $t_2 = w(n)$ we have

$$\varphi\left(\frac{x^*(\sigma^{-1}(n))}{w(n)}\right) w(n) + \varphi\left(\frac{x^*(n)}{w(\sigma(n))}\right) w(\sigma(n)) \ge \varphi\left(\frac{x^*(\sigma^{-1}(n))}{w(\sigma(n))}\right) w(\sigma(n)) + \varphi\left(\frac{x^*(n)}{w(n)}\right) w(n).$$

Hence if τ is the transposition which exchanges the indices n and $\sigma(n)$, that is if $\tau(i) = i$ for $i \neq n, \sigma(n)$, and $\tau(n) = \sigma(n)$, $\tau(\sigma(n)) = n$, then we get

$$\sum_{i=1}^{n} \varphi\left(\frac{x^*(i)}{w(\sigma(i))}\right) w(\sigma(i)) \geq \sum_{i=1}^{n} \varphi\left(\frac{x^*(i)}{w(\tau\circ\sigma(i))}\right) w(\tau\circ\sigma(i)).$$

Finally since the permutation $\tau\circ\sigma$ fixes the index n, that is $\tau\circ\sigma(n) = n$, we can apply the previous case and the induction is finished. $\square$

Corollary 2.2. *Let $w = \{w(i)\}_{i=1}^n$ be a finite, decreasing and positive sequence, and (a_{ij}) be a $n\times n$ matrix of non-negative numbers satisfying the condition*

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji} \qquad i = 1,\dots,n,$$

that is the i-th row and the i-th column have the same sum, then for every sequence of real numbers $\{x(i)\}_{i=1}^n$ we have

$$\sum_{i,j=1}^{n} \varphi\left(\frac{x^*(i)}{w(i)}\right) w(i)a_{ij} \leq \sum_{i,j=1}^{n} \varphi\left(\frac{x^*(i)}{w(j)}\right) w(j)a_{ij}.$$

Proof. We first reduce the proof to the case where a_{ij} are positive rational numbers. In fact we approximate a_{ij}, $1 \le i \le n, 1 \le j < n$, by rational numbers q_{ij} and then use the relations $\sum_j q_{ij} = \sum_j q_{ji}$ for defining the last row of the approximating matrix. Then by homogeneity we can further reduce it to the case where a_{ij} are natural numbers. We consider then a partition of the interval $J = [[1, \sum_{ij} a_{ij}]]$ into disjoint intervals $I_{ij} \subset \mathbb{N}$, with the respective lengths $|I_{ij}| = a_{ij}$. We put them in lexicographic order

$$I_{11} < I_{12} < \dots < I_{1n} < I_{21} < \dots < I_{2n} < \dots < I_{n1} < \dots < I_{nn},$$

and define $A_i = \bigcup_j I_{ij}$ and $B_j = \bigcup_i I_{ij}$. Set $\hat{x}(k) = x^*(i)$ and $\hat{w}(k) = w(i)$ for $k \in A_i$. Note that the sequences $\hat{x}$ and $\hat{w}$ are decreasing and $|A_i| = |B_i|$. So we can find a permutation σ of J mapping A_i onto B_i for each $i = 1,\dots,n$. Then the two sides of the desired inequality are respectively equal to $\sum_k \varphi\left(\frac{\hat{x}^*(k)}{\hat{w}(k)}\right)\hat{w}(k)$ and $\sum_k \varphi\left(\frac{\hat{x}^*(k)}{\hat{w}(\sigma(k))}\right)\hat{w}(\sigma(k))$. Consequently, by Proposition 2.1 the proof is finished. $\square$

Lemma 2.3. *Let $w = \{w(n)\}$ be a decreasing weight sequence. Then for every sequence x,*

$$\begin{aligned}
\sum_{n=1}^{\infty} \varphi\left(\frac{x^*(n)}{w(n)}\right) w(n) &\leq \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) v(n) : v \sim w, v \geq 0\right\}\\
&\leq \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) v(n) : v \sim w, v > 0\right\}\\
&\leq \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{w\circ\sigma(n)}\right) w\circ\sigma(n) : \sigma \textit{ automorphism of } \mathbb{N}\right\}.
\end{aligned}$$

Proof. Observe that the second inequality is obvious and the third one is also clear since for every automorphism σ of $\mathbb{N}$ we have $(w\circ\sigma)^* = w$.

In order to prove the first one, let $v \geq 0$ be any sequence such that $v \sim w$. We observe first that if $\operatorname{supp} w \neq \mathbb{N}$, then both sides of the first inequality are equal. In fact if $|\operatorname{supp} x| \leq |\operatorname{supp} w| < \infty$, then the situation is reduced to the finite case as in Proposition 2.1. If $|\operatorname{supp} x| > |\operatorname{supp} w|$ then $m(x) = \infty$, and for any $v \sim w$ there is at least one i such that $v(i) = 0$ and $x^*(i) > 0$. So the right side is also infinity. Thus we assume that $w > 0$.

We claim first that if $x(n) \nrightarrow 0$ as $n \to \infty$, then each expression above is equal to infinity and the equalities hold. Indeed if $x(n) \nrightarrow 0$ then $\inf_n x^*(n) = K > 0$. Since w is decreasing, there exists $b > 0$ such that the set $S = \{n : w(n) \le b\}$ is infinite. Hence using the fact that the function $t \mapsto \varphi(a/t)t$ is decreasing for a fixed a,

$$\sum_{n=1}^{\infty} \varphi\left(\frac{x^*(n)}{w(n)}\right) w(n) \ge \sum_{n\in S} \varphi\left(\frac{x^*(n)}{w(n)}\right) w(n) \ge \sum_{n\in S} \varphi\left(\frac{K}{b}\right) b = \infty.$$

Similarly, for any $v \sim w$, $v \ge 0$, the set $\{n : v(n) \le b\}$ is also infinite, and so $\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) v(n) = \infty$. Therefore all expressions in the above inequalities are equal to infinity.

Suppose now that $x(n) \to 0$ if $n \to \infty$. Letting $n \in \mathbb{N}$ be arbitrary, there exists an automorphism τ_n of $\mathbb{N}$ such that $x^*(i) = |x \circ \tau_n(i)|$ for all $i = 1, \dots, n$. Then $(v \circ \tau_n)^* = v^* = w$, and so setting a finite sequence $v_n = (v \circ \tau_n(i))_{i=1}^n$, we have that its rearrangement $v_n^*(i) \le w(i)$ for all $i = 1, \dots, n$. Assume $v_n^*(i) > 0$ for all $i \in \{1, \dots, n\}$. Then applying the fact that the function $t \mapsto \varphi(a/t)t$ is decreasing for any fixed $a > 0$, and Proposition 2.1, we get the following inequalities

$$\begin{aligned}\sum_{i=1}^{n} \varphi\left(\frac{x^*(i)}{w(i)}\right) w(i) &\le \sum_{i=1}^{n} \varphi\left(\frac{x^*(i)}{v_n^*(i)}\right) v_n^*(i) \le \sum_{i=1}^{n} \varphi\left(\frac{x^*(i)}{v\circ\tau_n(i)}\right) v\circ\tau_n(i)\\ &= \sum_{i=1}^{n} \varphi\left(\frac{|x\circ\tau_n(i)|}{v\circ\tau_n(i)}\right) v\circ\tau_n(i) \le \sum_{i=1}^{\infty} \varphi\left(\frac{|x(i)|}{v(i)}\right) v(i).\end{aligned}$$

Now let $v_n^*(i) = 0$ for some $i \in \{1, \dots, n\}$. Then if $x^*(i) = 0$, by our convention each i-th term of the first four sums above is equal to zero, so the inequalities hold. If $x^*(i) > 0$ then $\varphi(x^*(i)/v_n^*(i))v_n^*(i) = \infty$, and so the second sum is also infinity. By definition of τ_n there is $k \in \{1, \dots, n\}$ such that $|x \circ \tau_n(k)| > 0$ and $v \circ \tau_n(k) = 0$, so at least one of the term in the third sum is infinity. Finally the last sum is infinity since for $j = \tau_n(k)$ we have $|x(j)| > 0$ and $v(j) = 0$. Thus in all cases the inequalities are satisfied.

Letting then $n \to \infty$ we obtain that

$$\sum_{i=1}^{\infty} \varphi\left(\frac{x^*(i)}{w(i)}\right) w(i) \le \sum_{i=1}^{\infty} \varphi\left(\frac{|x(i)|}{v(i)}\right) v(i),$$

for every $v \ge 0$ and $v \sim w$. This allows us to pass to the infimum on the right side and the proof is completed.

□

Theorem 2.4. *Let $w = \{w(n)\}$ be a decreasing weight sequence. Then for any $x = \{x(n)\}$,*

$$\sum_{n=1}^{\infty} \varphi\left(\frac{x^*(n)}{w(n)}\right) w(n) = \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) v(n) : v \sim w, v \ge 0\right\}.$$

Proof. As in the proof of Lemma 2.3 we can assume that w is positive and $x(n) \to 0$ as $n \to \infty$. If $\operatorname{supp} x = \mathbb{N}$ or $|\operatorname{supp} x| < \infty$ then there exists a one-to-one and onto mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $|x(n)| = x^* \circ \sigma(n)$, $n \in \mathbb{N}$. If $|\operatorname{supp} x| = \infty$ but $\operatorname{supp} x \ne \mathbb{N}$, then we find $\sigma : \operatorname{supp} x \to \mathbb{N}$, one-to-one, onto and such that $|x(n)| = x^* \circ \sigma(n)$, $n \in \operatorname{supp} x$. In both cases

$$m(x) = \sum_{n\in \operatorname{supp} x} \varphi\left(\frac{|x(n)|}{w\circ\sigma(n)}\right) w\circ\sigma(n).$$

In the first case defining $v = w \circ \sigma$, we have $v \sim w$ and $v > 0$. In the second case define $v = \{v(n)\}$ such that $v(n) = w \circ \sigma(n)$ for $n \in \operatorname{supp} x$, and $v(n) = 0$ for $n \notin \operatorname{supp} x$. Since the range of σ is $\mathbb{N}$, it is clear that $v \sim w$, $v \ge 0$, and by our convention $m(x) = \sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) v(n)$, which completes the proof.

□

Theorem 2.5. *Let w be a decreasing weight function on I.*

(i) For every $f \in L^0$ we have

$$\int_I \varphi\left(\frac{f^*}{w}\right) w = \inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right) v : v \sim w, v \geq 0\right\}.$$

(ii) If $I = (0,a)$ with $a < \infty$ we also have

$$\begin{aligned}\int_I \varphi\left(\frac{f^*}{w}\right) w &= \inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right) v : v \sim w, v > 0\right\}\\ &= \inf\left\{\int_I \varphi\left(\frac{|f|}{w\circ\sigma}\right) w\circ\sigma : \sigma \text{ measure preserving mapping from } I \text{ onto } I\right\}.\end{aligned}$$

Proof. Similarly as in the proof of Lemma 2.3 we can assume that $w > 0$ on I. Assume that the left-hand side of the equality in (i) is finite. Then reasoning similarly as in the proof of Lemma 2.3, $\lim_{t\to\infty} f^*(t) = 0$ and so by Ryff's Theorem [1, Cor. 7.6 in Chap. 2] there exists a measure preserving and onto map $\tau : \operatorname{supp} f \to \operatorname{supp} f^*$ such that $|f| = f^* \circ \tau$ on $\operatorname{supp} f$. If $\operatorname{supp} f = I$ or $|\operatorname{supp} f| < \infty$ then τ is, resp. can be extended to a measure preserving mapping from I onto I. In the other case we have $\operatorname{supp} f^* = I$. Thus the range of τ is I in both cases. Hence letting $v(t) = w \circ \tau(t)$ for $t \in \operatorname{supp} f$ and $v(t) = 0$ for $t \notin \operatorname{supp} f$, we have $v \sim w$, $v \geq 0$, and by our convention

$$\int_I \varphi\left(\frac{f^*}{w}\right) w = \int_{\operatorname{supp} f} \varphi\left(\frac{f^*\circ\tau}{w\circ\tau}\right) w\circ\tau = \int_{\operatorname{supp} f} \varphi\left(\frac{|f|}{w\circ\tau}\right) w\circ\tau = \int_I \varphi\left(\frac{|f|}{v}\right) v.$$

Thus we get that the left-hand side of the equality in (a) is greater than the right-hand side.

For case (ii) observe that if $a < \infty$ then we can find a measure preserving transformation τ on I such that $|f| = f^* \circ \tau$. Then $v = w \circ \tau > 0$ and $v \sim w$. Thus

$$\begin{aligned}\int_I \varphi\left(\frac{f^*}{w}\right) w &= \int_I \varphi\left(\frac{f^*\circ\tau}{w\circ\tau}\right) w\circ\tau = \int_I \varphi\left(\frac{|f|}{w\circ\tau}\right) w\circ\tau\\ &\geq \inf\left\{\int_I \varphi\left(\frac{|f|}{w\circ\sigma}\right) w\circ\sigma : \sigma \text{ measure preserving mapping from } I \text{ to } I\right\}\\ &\geq \inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right) v : v \sim w, v > 0\right\}.\end{aligned}$$

As for the converse direction we need to prove that for every weight $v \sim w$ and $v \geq 0$, we have

$$\int_I \varphi\left(\frac{f^*}{w}\right) w \leq \int_I \varphi\left(\frac{|f|}{v}\right) v. \tag{2}$$

If there exists $A \subset I$ with $|A| > 0$ such that $f(t) \neq 0$ and $v(t) = 0$ a.e. on A, then (2) is satisfied since its right side is equal to infinity. Thus assume further that $\operatorname{supp} f \subset \operatorname{supp} v$. By standard approximation argument we may assume that f is a simple function, and consequently that f^* is a decreasing step function as below

$$f^* = \sum_{i=1}^N x_i^* \chi_{A_i}, \qquad |f| = \sum_{i=1}^N x_i^* \chi_{E_i}$$

with $x_i^* > 0$, $A_i = [a_{i-1}, a_i)$, $0 = a_0 < a_1 < \cdots < a_N < \infty$ and $|E_i| = |A_i| = a_i - a_{i-1}$, $i = 1, \ldots, N$. Let $A = [0, a_N)$, $E = \bigcup E_i$, $v_E = v\chi_E$ and $v_E^* = (v_E)^*$. By the assumption that $E = \operatorname{supp} f \subset \operatorname{supp} v$ we

have that $v_E > 0$ on E and $v_E^* > 0$ on A. Moreover $v_E^* \le v^* = w^*$. Hence

$$\int_I \varphi\left(\frac{f^*}{w}\right) w = \int_A \varphi\left(\frac{f^*}{w}\right) w \le \int_A \varphi\left(\frac{f^*}{v_E^*}\right) v_E^* = \int_I \varphi\left(\frac{f^*}{v_E^*}\right) v_E^*.$$

For each i there is a measure-preserving transformation $\tau_i : A_i \to E_i$. Let $\tau : A \to E$ be the measure preserving transformation defined by $\tau(t) = \tau_i(t)$ whenever $t \in A_i$. Then $|f| \circ \tau = f^*$, $v_E \circ \tau$ is a positive function on A with $(v_E \circ \tau)^* = v_E^*$ and

$$\int_I \varphi\left(\frac{|f|}{v}\right) v = \int_A \varphi\left(\frac{f^*}{v_E \circ \tau}\right) v_E \circ \tau.$$

Thus it is sufficient to prove (2) with $v_E^*, v_E \circ \tau, f^*, A$ respectively in place of w, v, f, I. Observe that both v_E^* and $v_E \circ \tau$ are positive on A. Therefore we can reduce the proof to the case where I has finite measure, f is a decreasing, positive simple function and v is positive on I. Now, another approximation argument allows to reduce to the case where w is also a simple function (approximating e.g. w by $w + \varepsilon$, $\varepsilon > 0$, and then $w + \varepsilon$ from below by simple decreasing functions $w_{n,\varepsilon} \ge \varepsilon$). Note that if $v \sim w$ then $v = w \circ \sigma$ for some measure preserving map $I \to I$, and v is approximated by the simple functions $w_{n,\varepsilon} \circ \sigma$. Also, refining the partition if necessary, we may suppose that the function w is built on the same intervals as the function $f^* = f$. Thus we may suppose $w = \sum_{i=1}^N w_i \chi_{A_i}$ and $v = \sum_{i=1}^N w_i \chi_{B_i}$ with $|B_i| = |A_i|$ and $w_i > 0$. Let $a_{ij} = |A_i \cap B_j|$. Since (A_i) and (B_i) are two partitions of the interval I we have $\sum_j a_{ij} = |A_i|$, $\sum_i a_{ij} = |B_j| = |A_j|$, so $\sum_j a_{ij} = \sum_j a_{ji}$ for all $i = 1, \dots, N$. Finally applying Corollary 2.2 we get

$$\int_I \varphi\left(\frac{f^*}{w}\right) w = \sum_{i,j=1}^N a_{ij} \varphi\left(\frac{x_i}{w_i}\right) w_i \le \sum_{i,j=1}^N a_{ij} \varphi\left(\frac{x_i}{w_j}\right) w_j = \int_I \varphi\left(\frac{f}{v}\right) v,$$

and the proof is completed. □

If we wish to approximate $m(x)$ in Theorem 2.4, or $M(f)$ in Theorem 2.5 for $a = \infty$, by the infimum over positive v or $w \circ \sigma$ where σ is an automorphism of $\mathbb{N}$ or measure preserving transformation of $(0, \infty)$, we need some additional assumptions on w and φ as we will see below. We start with a preparatory lemma.

Lemma 2.6. *Let w be a positive decreasing weight function on $(0, \infty)$. Let $f \in L^0$ be such that $lim_{t\to\infty} f^*(t) = 0$, $|\operatorname{supp} f| = \infty$ and $|(\operatorname{supp} f)^c| > 0$. Then for every $T > 0$ there exists a measure preserving and surjective mapping $\sigma : (0, \infty) \to (0, \infty)$ such that*

$$\int_0^\infty \varphi\left(\frac{|f|}{w \circ \sigma}\right) w \circ \sigma \le \int_0^T \varphi\left(\frac{f^*}{w}\right) w + \int_T^\infty \varphi\left(\frac{f^*(t)}{w(2t)}\right) w(2t)\, dt.$$

An analogous statement remains true in discrete case, where f is replaced by a sequence, σ by a one-to-one and onto mapping of $\mathbb{N}$, and integration by summation.

Proof. We shall prove this only in the case when $f \in L^0$ is such that $\lim_{t\to\infty} f^*(t) = 0$ and $|\operatorname{supp} f| = |(\operatorname{supp} f)^c| = \infty$. Then there exist $\tau : \operatorname{supp} f \to \operatorname{supp} f^* = (0, \infty)$ and $\rho : (\operatorname{supp} f)^c \to (0, \infty)$ such that both are measure preserving and onto, and moreover $|f| = f^* \circ \tau$. Let $T > 0$. Define

$$\sigma(t) = \begin{cases} \tau(t) + jT & \text{if } \ t \in \tau^{-1}(jT, (j+1)T), j = 0, 1, \dots; \\ \rho(t) + (j-1)T & \text{if } \ t \in \rho^{-1}(jT, (j+1)T), j = 1, 2, \dots. \end{cases}$$

It is easy to check that $\sigma : (0,\infty) \to (0,\infty)$ is measure preserving and onto. Then

$$\begin{aligned}\int_0^\infty \varphi\left(\frac{|f|}{w\circ\sigma}\right) w\circ\sigma &= \int_{\operatorname{supp} f} \varphi\left(\frac{|f|}{w\circ\sigma}\right) w\circ\sigma = \int_{\operatorname{supp} f} \varphi\left(\frac{f^*\circ\tau}{w\circ\sigma}\right) w\circ\sigma \\ &= \int_{\tau^{-1}(0,T)} \varphi\left(\frac{f^*\circ\tau}{w\circ\tau}\right) w\circ\tau + \sum_{j=1}^{\infty}\int_{\tau^{-1}(jT,(j+1)T)} \varphi\left(\frac{f^*(\tau(t))}{w(\tau(t)+jT)}\right) w(\tau(t)+jT)\,dt \\ &= \int_0^T \varphi\left(\frac{f^*}{w}\right) w + \sum_{j=1}^{\infty}\int_{jT}^{(j+1)T} \varphi\left(\frac{f^*(u)}{w(u+jT)}\right) w(u+jT)\,du.\end{aligned}$$

Now by decreasing monotonicity of the function $t \mapsto \varphi(c/t)t$ for any $c > 0$, we have that for $u \in (jT,(j+1)T)$ it holds

$$\varphi\left(\frac{f^*(u)}{w(u+jT)}\right) w(u+jT) \le \varphi\left(\frac{f^*(u)}{w(2u)}\right) w(2u).$$

Hence

$$\begin{aligned}\int_0^\infty \varphi\left(\frac{|f|}{w\circ\rho}\right) w\circ\rho &\le \int_0^T \varphi\left(\frac{f^*}{w}\right) w + \sum_{j=1}^{\infty}\int_{jT}^{(j+1)T} \varphi\left(\frac{f^*(u)}{w(2u)}\right) w(2u)\,du \\ &= \int_0^T \varphi\left(\frac{f^*}{w}\right) w + \int_T^\infty \varphi\left(\frac{f^*(u}{w(2u)}\right) w(2u)\,du,\end{aligned}$$

and the proof is completed.

□

Now we are ready to present more refined results than Theorems 2.4 and 2.5, but with an additional assumption of control of the decreasing slope of w in relation to φ.

Definition 2.7. A positive weight function (resp., sequence) w is said to be controlled by the Orlicz function φ (or shortly "to be *φ-controlled*") if for some $K > 0$ and every $c > 0$, $t \in I$ with $2t \in I$, (resp., every $n \in \mathbb{N}$) it holds

$$\varphi\left(\frac{c}{w(2t)}\right) w(2t) \le K\varphi\left(\frac{c}{w(t)}\right) w(t) \quad \left(\text{resp.,}\quad \varphi\left(\frac{c}{w(2n)}\right) w(2n) \le K\varphi\left(\frac{c}{w(n)}\right) w(n)\right).$$

Remark 2.8. (i) Trivial cases are the constant weights that are controlled by every Orlicz function, and the Orlicz function $\varphi(t) = t$ which controls every weight.

(ii) If φ and $1/w$ satisfy Δ_2 condition, then w is φ-controlled. Notice that if w is regular then $1/w$ satisfies Δ_2-condition. In fact, by regularity of w for some $C > 0$ and all $t \in I$, $W(t) \le Ctw(t)$. Hence $tw(t) \le W(2t) \le 2Ctw(2t)$ and so $1/w(2t) \le 2C/w(t)$ which means that $1/w$ satisfies Δ_2 condition.

Recall that the dilation operator D_2 is defined for $f \in L^0$ as $D_2 f(t) = f(t/2)$, $t \in I$, and for a sequence x as $D_2 x(n) = x(\lceil n/2 \rceil)$.

Theorem 2.9. *Let φ be an Orlicz function.* (i) *Let w be a φ-controlled decreasing weight function on $I = (0,\infty)$. Then for every $f \in L^0$,*

$$\begin{aligned}\int_0^\infty \varphi\left(\frac{f^*}{w}\right) w &= \inf\left\{\int_0^\infty \varphi\left(\frac{|f|}{v}\right) v : v \sim w, v > 0\right\} \\ &= \inf\left\{\int_0^\infty \varphi\left(\frac{|f|}{w\circ\sigma}\right) w\circ\sigma : \sigma \ \ \textit{measure preserving mapping from } I \textit{ to } I\right\}.\end{aligned}$$

(ii) *Let $w = \{w(n)\}$ be a φ-controlled decreasing weight sequence. Then for any $x = \{x(n)\}$,*

$$\sum_{n=1}^{\infty} \varphi\left(\frac{x^*(n)}{w(n)}\right) w(n) = \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) v(n) : v \sim w, v > 0\right\}$$
$$= \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{w\circ\sigma(n)}\right) w\circ\sigma(n) : \sigma \text{ automorphism of } \mathbb{N}\right\}.$$

If w is not necessarily φ-controlled then the above equalities hold true for those functions f or sequences x for which $M(D_2 f) < \infty$ or $m(D_2 x) < \infty$, respectively.

Proof. We shall only prove (i). By the proof of Theorem 2.5 the left-hand side of the first equality is less than its right-hand side, which in turn is less than the right-hand side of the second equality. For a converse it is enough to show that for any $\epsilon > 0$ there is a measure preserving and onto mapping σ of $(0,\infty)$ such that $\int_0^\infty \varphi\left(\frac{|f|}{w\circ\sigma}\right) w\circ\sigma \le M(f) + \epsilon$. Assume that $f \in L^0$ is such that

$$M(f) = \int_0^\infty \varphi\left(\frac{f^*}{w}\right) w < \infty.$$

It follows that $\lim_{t\to\infty} f^*(t) = 0$. If $|\operatorname{supp} f| < \infty$ or $\operatorname{supp} f = I$, then there exists a measure preserving transformation τ of I such that $|f| = f^*\circ\tau$. Hence $M(f) = \int_0^\infty \varphi\left(\frac{|f|}{w\circ\tau}\right) w\circ\tau$, and the equalities hold.

Suppose now that $|\operatorname{supp} f| = \infty$ and $|(\operatorname{supp} f)^c| > 0$. By the assumption that w is φ-controlled, there is $C > 0$ such that $\varphi\left(\frac{\alpha}{w(2t)}\right) w(2t) \le C\varphi\left(\frac{\alpha}{w(t)}\right) w(t)$ for every $\alpha, t > 0$. So

$$\int_0^\infty \varphi\left(\frac{f^*(t)}{w(2t)}\right) w(2t)\, dt \le CM(f) < \infty.$$

Then for any $\epsilon > 0$ there exists $T > 0$ such that

$$\int_T^\infty \varphi\left(\frac{f^*(t)}{w(2t)}\right) w(2t)\, dt < \epsilon.$$

An appeal to Lemma 2.6 gives now a suitable measure preserving mapping. □

The next result and its corollary provide another useful reformulations of $m(x)$ or $M(f)$.

Proposition 2.10. *Let w be a weight function. Then for every $f \in L^0$,*

$$\inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right) v : v^* \le w, v > 0\right\} = \inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right) v : v \sim w, v > 0\right\}.$$

Let $w = \{w(n)\}$ be a weight sequence. Then for every sequence $x = \{x(n)\}$,

$$\inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) v(n) : v^* \le w, v > 0\right\} = \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) v(n) : v \sim w, v > 0\right\}.$$

In the above formulas we can replace simultaneously in both sides $v > 0$ by $v \ge 0$.

Proof. We shall prove it only for the function case, assuming $I = (0,\infty)$. Since for every $v \sim w$, we have $v^* = w$, the left side is less than the right side. In the opposite direction we assume that the left side is finite. Consider any measurable positive function $k : I \to (0,\infty)$ with $k^* \le w$ and $\int \varphi(|f|/k)k < \infty$.

Let $\alpha = \lim_{t\to\infty} k^*(t)$ and $A = \{t : k(t) > \alpha\}$. Consider two cases.

Let first $|A| = \infty$. Then $((k-\alpha)\chi_A)^* = k^* - \alpha$ on I, and $\operatorname{supp}(k-\alpha)\chi_A = A$. By Ryff's theorem there exists an onto measure preserving transformation $\tau : A \to I$ such that for $t \in A$, $(k-\alpha)\chi_A(t) =$

$(k-\alpha)^* \circ \tau(t)$. Hence for $t \in A$, $k(t) - \alpha = k^* \circ \tau(t) - \alpha$, and so $k(t) = k^* \circ \tau(t)$ on A. Define $v(t) = w \circ \tau(t)$ if $t \in A$, and $v(t) = k(t)$ if $t \notin A$. Since $k^* \leq w$, so for $t \in A$, $k(t) = k^* \circ \tau(t) \leq w \circ (t) = v(t)$. Thus $k \leq v$ on I. Moreover, since for $t \in A$, $v(t) = w \circ \tau(t) \geq k(t) > \alpha$, and for $t \notin A$, $v(t) \leq \alpha$, in view of $|A| = \infty$, we have that for $t \in I$, $v^*(t) = (v\chi_A)^*(t) = ((w \circ \tau)\chi_A)^*(t) = w(t)$.

Let now $|A| < \infty$. Then let $B = \{t : k(t) = \alpha\}$. Since $\lim_{t\to\infty} k^*(t) = \alpha$, we have $|B| = \infty$. Then there exists $\tau : A \to \{t : k^*(t) > \alpha\}$ and $\tau : B \to \{t : k^* = \alpha\}$, such that $\tau : A \cup B \to I$ is onto and measure preserving. Moreover $k(t) = k^* \circ \tau(t)$ for $t \in A \cup B$. Let now $v(t) = w \circ \tau(t)$ for $t \in A \cup B$, and $v(t) = k(t)$ otherwise. Clearly $v \geq k$ on I. In view of $v(t) \geq \alpha$ on $A \cup B$ and $|A \cup B| = \infty$, we have that $(v\chi_{A\cup B})^* = v^*$. Hence $v^* = ((w \circ \tau)\chi_{A\cup B})^* = w$.

In both cases we have found v such that $v^* = w$ and $v \geq k$. Hence

$$\int_I \varphi\left(\frac{|f|}{v}\right) v \leq \int_I \varphi\left(\frac{|f|}{k}\right) k.$$

We complete the proof taking infimum first with respect to $v \sim w$ and then k such that $k^* \leq w$. $\square$

Corollary 2.11. *For any function $f \in L^0$,*

$$M(f) = \int_I \varphi\left(\frac{f^*}{w}\right) w = \inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right) v : v^* \leq w, v \geq 0\right\},$$

and for any sequence $x = \{x(n)\}$,

$$m(x) = \sum_{n=1}^{\infty} \varphi\left(\frac{x^*(n)}{w(n)}\right) w(n) = \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right) : v^* \leq w, v \geq 0\right\}.$$

3. The classes $M_{\varphi,w}$

In this section we investigate several aspects of the classes $M_{\varphi,w}$, which were defined in section 1: linear structure, concavity and Köthe duality.

3.1. **Linear structure.** It is known from the general theory [9, Lemma 1.4] that $M_{\varphi,w}$, resp. $m_{\varphi,w}$, is a linear space and the corresponding homogeneous functional $\|\cdot\|_M$, resp. $\|\cdot\|_m$, is a quasi-norm if and only if the dilation operator D_2 on the class $M_{\varphi,w}$, resp. $m_{\varphi,w}$, is bounded. In this case $M_{\varphi,w}$, resp. $m_{\varphi,w}$ are complete, and thus quasi-Banach spaces. In view of [9, Proposition 4.5] a simple sufficient condition for boundedness of D_2 on $M_{\varphi,w}$ is the following inequality

$$\varphi\left(\frac{c}{w(2t)}\right) w(2t) \leq \varphi\left(\frac{Cc}{w(t)}\right) w(t) \tag{3}$$

for some $C > 0$ and all $c > 0$, $t > 0$ with $2t \in I$. By the fact that $t \mapsto \varphi(c/t)t$ is decreasing for $t > 0$ and any $c > 0$, it holds in particular if $1/w$ verifies condition Δ_2 (regardless of φ). Recall that any function $h : I \to \mathbb{R}_+$ satisfies condition Δ_2 if $h(2t) \leq Kh(t)$ for some $K > 0$ and all $t \in I$ such that $2t \in I$.

The class $M_{\varphi,w}$ cannot be linear when D_2 is not bounded on it. More precisely we have the following result.

Lemma 3.1. *The following assertions are equivalent.*

(i) *The dilation operator D_2 acts on the class $M_{\varphi,w}$.*
(ii) *The dilation operator D_2 acts and is bounded on the class $M_{\varphi,w}$.*
(iii) *The class $M_{\varphi,w}$ is linear.*
(iv) *The class $M_{\varphi,w}$ is linear and $\|\cdot\|_M$ is a quasi-norm.*

A trivial, but useful fact which will be used in the proof of this lemma is that the cones of decreasing, nonnegative functions in $M_{\varphi,w}$ and in the Banach function space $\mathcal{L}_{\varphi,w}$ coincide, where $\mathcal{L}_{\varphi,w} = \{f \in L^0 : \|f\|_{\mathcal{L}_{\varphi,w}} := \inf\{\epsilon > 0 : \int_I \varphi(|f|/(\epsilon w))w \leq 1\} < \infty\}$, and the homogeneous functional $\|\cdot\|_M$ coincides with the norm $\|\cdot\|_{\mathcal{L}_{\varphi,w}}$ on this cone. In particular this cone is σ-convex that is closed under infinite convex combinations, and the restriction of the functional $\|\cdot\|_M$ to this cone is σ-convex.

Proof of Lemma 3.1. (i) $\implies$ (iii) and (ii) $\implies$ (iv) follow from the classical inequality $(f+g)^* \leq D_2 f^* + D_2 g^*$. (iii) $\implies$ (i) and (iv) $\implies$ (ii) follow from the equality $(f_1+f_2)^* = D_2 f^*$ whenever f_1 and f_2 are two disjoint functions both of which are equimeasurable with f.

Clearly (ii) implies (i). Assume that (i) holds true but not (ii). There exists a sequence (f_n) in $M_{\varphi,w}$ with $\|D_2 f_n\|_M > 4^n \|f_n\|_M$ and $\|f_n\|_M \leq 1$ for all $n \in \mathbb{N}$. We may assume w.l.o.g. that $f_n = f_n^*$ and $\|f_n\|_M = 1$ for every $n \geq 1$. Set $f = \sum_{n=1}^\infty 2^{-n} f_n$. Since the cone of positive decreasing functions in $M_{\varphi,w}$ is σ-convex, it holds that $f \in M_{\varphi,w}$. Note that $D_2 f \geq 2^{-n} D_2 f_n$ for every $n \geq 1$. Thus if $D_2 f \in M_{\varphi,w}$ we obtain $\|D_2 f\|_M \geq 2^{-n}\|D_2 f_n\|_M \geq 2^{-n} \times 4^n = 2^n$ for every $n \geq 1$, a contradiction. Hence (i) implies (ii). $\square$

Denote by F_M the fundamental function of $M_{\varphi,w}$, defined as usual by

$$F_M(t) = \|\chi_{(0,t)}\|_M, \quad t \in I.$$

Note that for every $t > 0$ such that $2t \in I$, $D_2 \chi_{[0,t]} = \chi_{[0,2t]}$. Thus D_2 is bounded on characteristic functions if and only if the fundamental function F_M verifies the condition Δ_2. In particular if F_M does not verify the condition Δ_2 then the class $M_{\varphi,w}$ is not linear. Below we will give a partial converse of that statement. But let us first give a simple explicit criterion for the fundamental function F_M to verify condition Δ_2. For $t \in I$ we set

$$G_M(t) = \frac{1}{w(t)\varphi^{-1}\left(\frac{1}{tw(t)}\right)}.$$

Note that the function G_M is increasing. Indeed if $0 < s \leq t \in I$ then

$$w(t)\varphi^{-1}\left(\frac{1}{tw(t)}\right) \leq w(t)\varphi^{-1}\left(\frac{1}{sw(t)}\right) \leq w(s)\varphi^{-1}\left(\frac{1}{sw(s)}\right)$$

since the function $u \mapsto \varphi^{-1}(cu)/u$ is decreasing on $(0,\infty)$ for any $c > 0$.

Lemma 3.2. *The fundamental function F_M verifies the condition Δ_2 if and only if the function G_M does. Moreover if this is the case, then the functions F_M and G_M are equivalent.*

Proof. Indeed, since for every $c > 0$ the function $s \mapsto \varphi\left(\frac{c}{w(s)}\right) w(s)$ is increasing, we have for $t \in I$,

$$1 = \int_0^t \varphi\left(\frac{1}{F_M(t)w(s)}\right) w(s)ds \leq t\varphi\left(\frac{1}{F_M(t)w(t)}\right) w(t),$$

and similarly

$$1 = \int_0^{2t} \varphi\left(\frac{1}{F_M(2t)w(s)}\right) w(s)ds \geq \int_t^{2t} \varphi\left(\frac{1}{F_M(2t)w(s)}\right) w(s)ds \geq t\varphi\left(\frac{1}{F_M(2t)w(t)}\right) w(t).$$

Hence for $t > 0$ such that $2t \in I$,

$$F_M(t) \leq G_M(t) \leq F_M(2t). \tag{4}$$

Thus if F_M verifies condition Δ_2, then F_M and G_M are equivalent and G_M must also satisfy condition Δ_2. Moreover rewriting (4) as

$$G_M(t/2) \leq F_M(t) \leq G_M(t)$$

we see that if G_M verifies Δ_2 then F_M and G_M are equivalent and F_M verifies also condition Δ_2. $\square$

Recall now the lower and upper (Matuszewska-Orlicz) indices [16, 1, 13] for a function $h : I \to \mathbb{R}_+$,

$$\alpha_h = \sup\{p \in \mathbb{R} : \exists C > 0\ \forall t \in I\ \forall 0 < \lambda \leq 1\ \ h(\lambda t) \leq C\lambda^p h(t)\},$$
$$\beta_h = \inf\{p \in \mathbb{R} : \exists C > 0\ \forall t \in I\ \forall 0 < \lambda \leq 1\ \ h(\lambda t) \geq C\lambda^p h(t)\}.$$

Clearly the indices are preserved by equivalent functions. Notice that for Orlicz function φ we have $I = \mathbb{R}_+$ in the definitions of indices.

Lemma 3.3. *Assume that $\alpha_\varphi > 1$. Then the fundamental function of $M_{\varphi,w}$ satisfies condition Δ_2 if and only if $1/w$ does.*

Proof. In view of Lemma 3.2 we have only to prove that G_M verifies Δ_2 if and only $1/w$ does.

If $1/w$ satisfies Δ_2, that is $C = \sup\limits_{t\in I} \frac{w(t)}{w(2t)} < \infty$ then using the fact that w is decreasing and φ^{-1} is concave we get for $t \in I$,

$$G_M(2t) = \frac{\frac{1}{w(2t)}}{\varphi^{-1}(\frac{1}{2tw(2t)})} \leq \frac{\frac{C}{w(t)}}{\varphi^{-1}(\frac{1}{2tw(t)})} \leq \frac{\frac{C}{w(t)}}{\frac{1}{2}\varphi^{-1}(\frac{1}{tw(t)})} = 2CG_M(t),$$

and so G_M fulfils Δ_2 condition.

Assume now that G_M verifies the condition Δ_2 and set $C = \sup\limits_{t\in I} \frac{G_M(2t)}{G_M(t)} < \infty$. By hypothesis we have $\alpha_\varphi > 1$, which implies that $\beta_{\varphi^{-1}} = 1/\alpha_\varphi < 1$, that is for some $p < 1$, $d \geq 1$, and all $u > 0$, $\lambda \geq 1$ we have $\varphi^{-1}(\lambda u) \leq d\lambda^p\varphi^{-1}(u)$. Then for $2t \in I$ we have

$$CG_M(t) \geq G_M(2t) \geq \frac{\frac{1}{w(2t)}}{d\left(\frac{w(t)}{w(2t)}\right)^p \varphi^{-1}\left(\frac{1}{2tw(t)}\right)} = \frac{1}{d}\left(\frac{w(t)}{w(2t)}\right)^{1-p} \frac{\frac{1}{w(t)}}{\varphi^{-1}\left(\frac{1}{2tw(t)}\right)} \geq \frac{1}{d}\left(\frac{w(t)}{w(2t)}\right)^{1-p} G_M(t).$$

Hence $w(t)/w(2t) \leq (dC)^{\frac{1}{1-p}}$ that is $1/w$ satisfies condition Δ_2. $\square$

Summing up the preceding results, we obtain a nice characterization of the normability for $M_{\varphi,w}$ classes, at least when φ has convexity "better than 1".

Proposition 3.4. *Assume that $\alpha_\varphi > 1$. Then the following assertions are equivalent.*

(i) *$M_{\varphi,w}$ is a linear space.*
(ii) *$M_{\varphi,w}$ is a linear space, and $\|\cdot\|_M$ is a quasinorm.*
(iii) *The fundamental function of $M_{\varphi,w}$ satisfies condition Δ_2.*
(iii) *$1/w$ satisfies condition Δ_2.*

3.2. **Concavity.** We draw now some consequences of section 2 on the geometry of spaces $M_{\varphi,w}$. We say that an Orlicz function φ is p-concave for some $1 < p < \infty$ if the map $t \mapsto \varphi(t^{1/p})$ on $\mathbb{R}_+$ is concave. Similarly a modular $G : L^0 \to \mathbb{R}_+$ is p-concave if the map $f \mapsto G(f^{1/p})$ is concave on $L_0^+(I)$.

Proposition 3.5. (1) The modular M is disjointly superadditive, that is $M(f+g) \geq M(f) + M(g)$ whenever $f, g \in L^0$ are disjoint.
(2) If the Orlicz function φ is p-concave for some $1 < p < \infty$ then so is the modular M.

Proof. (1) results from the fact that by Theorem 2.5 the modular M is the infimum of the disjointly additive maps $I_v : f \mapsto \displaystyle\int_I \varphi\left(\frac{|f|}{v}\right) v$, where $v \geq 0, v \sim w$.

Similarly (2) follows from the fact that the modulars I_v are clearly p-concave whenever φ is p-concave, and an infimum of p-concave modulars is p-concave. $\square$

Corollary 3.6. *If the Orlicz function φ is equivalent to a p-concave Orlicz function for some $1 < p < \infty$, then the class $M_{\varphi,w}$ is p-concave, that is for some constant $c > 0$, for all $f_1, \dots, f_n \in M_{\varphi,w}$ and every $n \in \mathbb{N}$,*

$$\left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_M \geq c \left(\sum_{i=1}^n \|f_i\|_M^p \right)^{1/p}.$$

Proof. Let $|g|^p = \sum_{i=1}^n |f_i|^p$. It is easy to check that if Orlicz functions are equivalent then the classes $M_{\varphi_1,w}$ and $M_{\varphi_2,w}$ coincide and $\|\cdot\|_{M_{\varphi_1,w}}$ and $\|\cdot\|_{M_{\varphi_2,w}}$ are equivalent. Thus we may assume w.l.o.g. that φ is p-concave, and that $\sum_{i=1}^n \|f_i\|_M^p = 1$. Let $c_i = \|f_i\|_M$ and $g_i = c_i^{-1} f_i$. We have $|g|^p = \sum_{i=1}^n c_i^p |g_i|^p$ with $\sum_{i=1}^n c_i^p = 1$. Denote $G(f) = M(|f|^p)$ and notice that $M(tg_i) > 1$ for $t > 1$. Then by concavity of G we get for all $t > 1$,

$$M(tg) = G(t^p |g|^p) \geq \sum_{i=1}^n c_i^p G(t^p |g_i|^p) = \sum_{i=1}^n c_i^p M(tg_i) > \sum_{i=1}^n c_i^p = 1,$$

and thus $\|g\|_M \geq 1$. □

3.3. **Köthe duality.** By *Köthe dual* of a subset A of L^0 we mean the set of elements f of L^0 such that $fg \in L_1$ for any $g \in A$, where as usual L_1 denotes the space of integrable functions on I equipped with the norm $\|f\|_1 = \int_I |f|$. We denote the Köthe dual of A by A'. Note that A' is a solid linear subspace in $L_0(I)$. In this subsection we shall determine the Köthe dual $M'_{\varphi,w}$ of the class $M_{\varphi,w}$. We do not assume this class to be quasi-normed nor even linear.

Lemma 3.7. *Every element of $M'_{\varphi,w}$ has a decreasing rearrangement which belongs also to $M'_{\varphi,w}$.*

Proof. Letting $f \in M'_{\varphi,w}$, we may assume w.l.o.g. that $f \geq 0$.

i) We show first that f has a finite rearrangement. If it is not the case, then for all $\lambda > 0$, $|\{t : f(t) > \lambda\}| = \infty$. Then we can find recursively a sequence $(A_n)_{n=1}^\infty$ of disjoint measurable sets of measure 1, such that for every $n \in \mathbb{N}$ we have $A_n \subset \{t : f(t) > 2^n F_M(n)\}$. Then setting

$$g = \sum_{n=1}^\infty \frac{2^{-n}}{F_M(n)} \chi_{A_n},$$

we get

$$g^* = \sum_{n=1}^\infty \frac{2^{-n}}{F_M(n)} \chi_{[n-1,n]} \leq \sum_{n=1}^\infty \frac{2^{-n}}{F_M(n)} \chi_{[0,n]} := h.$$

Clearly $M(h) \leq \sum_{n=1}^\infty 2^{-n} = 1$, and so $M(g) = M(g^*) \leq 1$. On the other hand

$$\int_I fg = \sum_{n=1}^\infty \int_{A_n} \frac{2^{-n}}{F_M(n)} f(t) dt = \infty,$$

a contradiction.

ii) Thus $f^*(t) < \infty$ for all $t \in I$. Set for every $n \in \mathbb{Z}$,

$$B_n = \{t \in I : 2^n \leq f < 2^{n+1}\} \text{ and } f_1 = \sum_{n \in \mathbb{Z}} 2^n \chi_{B_n}.$$

Then $f_1 \leq f \leq 2f_1$, and it is sufficient to prove that $f_1^* \in M'_{\varphi,w}$. Since the set of values of f_1 has zero as a unique possible accumulation point, we have

$$f_1^* = \sum_{n \in J} 2^n \chi_{I_n},$$

where J is an interval of $\mathbb{Z}$ and I_n an interval of I with $|I_n| = |B_n|$ for every $n \in J$. The first element in J is the supremum of those n that I_n has infinite measure, and the last element of J is the supremum of all integers n such that I_n has positive finite measure. In particular if for each $n \in \mathbb{Z}$, $0 < |B_n| < \infty$, then $\inf J = -\infty$ and $\sup J = \infty$, and then $f_1^* = \sum_{n=-\infty}^{\infty} 2^n \chi_{I_n}$.

Let $B = \bigcup_{n \in J} B_n$ and $f_2 = \chi_B f_1$. Then $f_1^* = f_2^*$ and there is a measure preserving transformation and onto $\sigma : B \to S$, where S is the support of f_1^*, such that $f_2 = f_1^* \circ \sigma$. If $g \in M_{\varphi,w}$, we have

$$\int f_1^*|g| = \int (f_1^* \circ \sigma)(|g| \circ \sigma) = \int f_2(|g| \circ \sigma).$$

This last integral is finite since $0 \le f_2 \le f \in M'_{\varphi,w}$ and $(g \circ \sigma)^* = (g\chi_S)^* \le g^* \in M_{\varphi,w}$. It follows that $f_1^* \in M'_{\varphi,w}$ and the proof is completed. □

Lemma 3.8. *For every $f \in M'_{\varphi,w}$ it holds that* $\sup\{\|fg\|_1 : g \in M_{\varphi,w}, \|g\|_M \le 1\} < \infty$.

Proof. By the ordinary Hardy-Littlewood inequality and the fact that f^* belongs to $M'_{\varphi,w}$ too, we may w.l.o.g. suppose that f is non-negative and decreasing and show that

$$\sup\{\|fg\|_1 : g \ge 0, \text{ decreasing}, \|g\|_M \le 1\} < \infty.$$

If it is not the case, we may find a sequence (g_n) of decreasing functions in $M_{\varphi,w}$, with $\|g_n\|_M \le 1$ and $\int_I f g_n \ge 2^n$, $n \in \mathbb{N}$. Setting $g = \sum_{n=1}^{\infty} 2^{-n} g_n$ we have that $g \ge 0$, g is decreasing and

$$\|g\|_M \le \sum_{n=1}^{\infty} 2^{-n} \|g_n\|_M \le 1,$$

since $\| \ \|_M$ is convex on the cone of non-negative decreasing elements of $M_{\varphi,w}$. On the other hand

$$\int_I fg = \sum_{n=1}^{\infty} 2^{-n} \int_I f g_n \ge \sum_{n=1}^{\infty} 2^{-n} \times 2^n = \infty,$$

a contradiction. □

Given a weight function w, by $L_\varphi(w)$ denote the Orlicz space associated to the Orlicz function φ and the measure wdt, that is $L_\varphi(w) = \{f \in L^0 : \exists \lambda > 0 \ \int_I \varphi(\lambda|f|)w < \infty\}$. It is equipped with the Luxemburg norm $\|f\|_{L_\varphi(w)} = \inf\{\epsilon > 0 : \int_I \varphi(|f|/\epsilon)w \le 1\}$. Recall also that the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ consists of all $f \in L^0$ such that $f^* \in L_\varphi(w)$.

Definition 3.9. Let φ be N-function, and φ_* be its complementary function. For $f \in \Lambda_{\varphi_*,w}$ we denote by $\|f\|^0_{\Lambda_{\varphi_*,w}}$ the Orlicz norm of f^* in the weighted Orlicz space $L_{\varphi_*}(w)$, that is

$$\|f\|^0_{\Lambda_{\varphi_*,w}} = \|f^*\|^0_{L_{\varphi_*}(w)} = \sup\left\{\int_I f^* g w : g \in L_\varphi(w), \|g\|_{L_\varphi(w)} \le 1\right\}.$$

We call $\|f\|^0_{\Lambda_{\varphi_*,w}}$ the *Orlicz norm* of f in $\Lambda_{\varphi_*,w}$ and by $\Lambda^0_{\varphi_*,w}$ denote the space $\Lambda_{\varphi_*,w}$ equipped with the Orlicz norm. Recall that the Orlicz norm in $L_{\varphi_*}(w)$ can be equivalently expressed by the Amemiya formula (e.g. [11, p. 92, Theorem 10.2] or [15, p. 7]) which gives here

$$\|f\|^0_{\Lambda_{\varphi_*,w}} = \|f^*\|^0_{L_{\varphi_*}(w)} = \inf_{k>0} \frac{1}{k}\left(1 + \int_I \varphi_*(kf^*)w\right).$$

For more information on Orlicz spaces we refer to [1, 12, 11, 13] and for Orlicz-Lorentz spaces to [9] and references there.

Theorem 3.10. *Let φ be N-function, and φ_* be its complementary function. Then $M'_{\varphi,w} = \Lambda^0_{\varphi_*,w}$ that is the Köthe dual of the class $M_{\varphi,w}$ is the classical Orlicz-Lorentz space $\Lambda_{\varphi_*,w}$, and the Orlicz norm on $\Lambda_{\varphi_*,w}$ is dual to the homogeneous functional $\|\cdot\|_M$ in the sense that $\|f\|^0_{\Lambda_{\varphi_*,w}} = \sup\left\{\int_I fg : g \in M_{\varphi,w}, \|g\|_M \le 1\right\}$, for all $f \in \Lambda_{\varphi_*,w}$.*

Remark 3.11. When $W(t) = \int_0^t w(s)\,ds = \infty$ for $t > 0$ the theorem states simply that $M'_{\varphi,w} = \{0\}$.

Proof. i) We prove first that $\Lambda_{\varphi_*,w} \subset M'_{\varphi,w}$, and more precisely

$$f \in \Lambda_{\varphi_*,w}, g \in M_{\varphi,w} \implies fg \in L_1 \text{ and } \|fg\|_1 \le \|f\|^0_{\Lambda_{\varphi_*,w}} \|g\|_M. \tag{5}$$

We have $f^* \in L_{\varphi_*}(w)$ while $g^*/w \in L_\varphi(w)$. Using the Hardy-Littlewood inequality and the Köthe duality of $(L_{\varphi_*}(w))' = L_\varphi(w)$ with respect to the new measure wdt, we have

$$\int_I |fg| \le \int_I f^* g^* = \int_I f^*(g^*/w)w \le \|f^*\|^0_{L_{\varphi_*}(w)} \|g^*/w\|_{L_\varphi(w)} = \|f\|^0_{\Lambda_{\varphi_*,w}} \|g\|_M.$$

ii) We prove now that the "unit ball" of $M_{\varphi,w}$ is 1-norming for $\Lambda_{\varphi_*,w}$ and its Orlicz norm, that is for all $f \in \Lambda_{\varphi_*,w}$,

$$\|f\|^0_{\Lambda_{\varphi_*,w}} = \sup\left\{\int_I fg : g \in M_{\varphi,w} : \|g\|_M \le 1\right\}. \tag{6}$$

We prove the statement (6) first when f is non-negative and decreasing. Since then $\|f\|^0_{\Lambda_{\varphi_*,w}} = \|f\|^0_{L_{\varphi_*}(w)}$, we can find for every $\varepsilon > 0$, a non-negative $h \in L_\varphi(w)$ with the Luxemburg norm $\|h\|_{L_\varphi(w)} \le 1$ such that

$$\int_I fhw \ge (1-\varepsilon)\|f\|^0_{\Lambda_{\varphi_*,w}}.$$

Let us show that h may be chosen to be decreasing. Note first that this does not result directly from the Hardy-Littlewood inequality because f^* is perhaps not equimeasurable with f for the measure wdt. We may assume that $W(t) < \infty$ for all $t > 0$, since otherwise $\Lambda_{\varphi_*,w} = \{0\}$ and the statement is trivial. Then with $J = W(I) = (0, b)$ for some $b \in (0, \infty]$,

$$\int_I fhw\,dt = \int_J (f \circ W^{-1})(h \circ W^{-1})du \le \int_J (f \circ W^{-1})(h \circ W^{-1})^* du = \int_I fh_1 w du,$$

where $h_1 = (h \circ W^{-1})^* \circ W$ is decreasing and equimeasurable with h for the measure wdt. In particular $\|h_1\|_{L_\varphi(w)} = \|h\|_{L_\varphi(w)} \le 1$. We put now $g = h_1 w$. Then g is decreasing and we have

$$\int_I fg \ge (1-\varepsilon)\|f\|^0_{\Lambda_{\varphi_*,w}},$$

$$\int_I \varphi\left(\frac{g}{w}\right) w = \int_I \varphi(h) w\,dt \le 1,$$

which implies that $\|g\|_M \le 1$. It shows (6) in the case when f is non-negative and decreasing.

We deduce now the statement (6) for general f. Since $\Lambda_{\varphi_*,w}$ with Orlicz norm has the Fatou property it is sufficient to prove the statement when f is a simple function with support of finite measure. In this case there is an invertible measure preserving automorphism of I such that $|f| = f^* \circ \sigma$. Then for any $g \in M_{\varphi,w}$ we have

$$\int_I f(\operatorname{sign} f)(g \circ \sigma) = \int_I |f|(g \circ \sigma) = \int_I (f^* \circ \sigma)(g \circ \sigma) = \int_I f^* g \text{ and } \|(\operatorname{sign} f) g \circ \sigma\|_M = \|g\|_M.$$

Therefore the statement results from the decreasing case.

iii) We show finally that $M'_{\varphi,w} \subset \Lambda_{\varphi_*,w}$.

Consider first the case where $W(t) < \infty$, for all $t \in I$. In this case $\Lambda_{\varphi_*,w}$ is not trivial and contains the bounded functions with support of finite measure. Let $f \in M'_{\varphi,w}$. Consider any bounded function h with support of finite measure such that $|h| \leq |f|$. We have $h \in \Lambda_{\varphi_*,w}$ and by (6) and Lemma 3.8 we get

$$\begin{aligned}\|h\|^0_{\Lambda_{\varphi_*,w}} &= \sup\left\{\int_I |hg| : g \in M_{\varphi,w} : \|g\|_M \leq 1\right\} \\ &\leq \sup\left\{\int_I |fg| : g \in M_{\varphi,w} : \|g\|_M \leq 1\right\} := C(f) < \infty.\end{aligned}$$

Since $|f| = \sup\{h \in L^0 : 0 \leq h \leq |f|, h$ bounded with finite measure support$\}$, and $\Lambda_{\varphi_*,w}$ has the Fatou property it results that $f \in \Lambda_{\varphi_*,w}$.

Consider now the case where $W(t) = \infty$, $t \in I$. In this case $\Lambda_{\varphi_*,w} = \{0\}$ and we have to prove that $M'_{\varphi,w}$ is trivial too. It is sufficient to prove that this space contains no indicator function, and by Lemma 3.7 that it does not contain $\chi_{[0,b]}$, $0 < b \in I$. In order to do this we test this function on the functions $f_u = (w \wedge w(u))\chi_{[0,b]}$, $u \in (0, b]$. For $c \geq 0$ we have

$$\begin{aligned}\int_0^b \varphi(cf_u/w)w &= \int_0^u \varphi(cw(u)/w(t))w(t)\,dt + \varphi(c)\int_u^b w(t)\,dt \\ &\leq \varphi(c)uw(u) + \varphi(c)\int_u^b w(t)\,dt = \varphi(c)\int_0^b w(t) \wedge w(u)\,dt.\end{aligned}$$

Thus choosing $c_u = \varphi^{-1}\left(\frac{1}{\int_0^b w(t)\wedge w(u)\,dt}\right)$ we have $\|c_u f_u\|_M \leq 1$. Note that $c_u \to 0$ when $u \to 0$. Then

$$\int_I \chi_{[0,b]} c_u f_u = c_u \int_0^b w(t) \wedge w(u)\,dt = \frac{c_u}{\varphi(c_u)} \to \infty \text{ when } u \to 0,$$

since φ is a N-function. This implies $\chi_{[0,b]} \notin M'_{\varphi,w}$ by Lemma 3.8 and completes the proof. □

Example 3.12. Here is an example of a decreasing function w such that $W(t) < \infty$, $t \in I$ but $1/w$ does not verify the condition Δ_2. Consequently if $\alpha_\varphi > 1$ the class $M_{\varphi,w}$ is not linear but its Köthe dual is non trivial. Let $I = (0, 1]$ and w be defined by $w(t) = 2^{k^2}$ when $t \in (4^{-(k+1)^2}, 4^{-k^2}]$, $k = 0, 1, \ldots$. Then $w(t) \leq t^{-1/2}$ for all $t \in I$ and so W is finite on I which implies that $M'_{\varphi,w} = \Lambda_{\varphi_*,w} \neq \{0\}$. Moreover $w(2t_k) = 2^{1-2k} w(t_k)$ for $t_k = 4^{-k^2}$, $k = 1, 2, \ldots$, which implies that $1/w$ does not verify Δ_2 condition.

4. A new class of Orlicz-Lorentz Spaces

It is well know that $\|\cdot\|_M$ (resp., $\|\cdot\|_m$) is a quasi-norm if the weight w is regular. Here we will show more. If w is regular then we will define explicitly a norm on $M_{\varphi,w}$ equivalent to $\|\cdot\|_M$ in function case or a norm on $m_{\varphi,w}$ equivalent to $\|\cdot\|_m$ in sequence case. In fact we will define a new class of r.i. Banach spaces induced by an Orlicz function φ and a weight w. It will turn out that these spaces are in fact the Köthe biduals of the classes $M_{\varphi,w}$.

4.1. **Definition and properties.** The formulas in Corollary 2.11 suggest the following definition.

Definition 4.1. Let φ be an Orlicz function and w be a positive weight sequence or a weight function. Define the following functionals

$$P(f) = \inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right)v : v \prec w, v \geq 0\right\}, \qquad f \in L^0,$$

$$p(x) = \inf\left\{\sum_{n=1}^{\infty} \varphi\left(\frac{|x(n)|}{v(n)}\right)v(n) : v \prec w, v \geq 0\right\}, \qquad x = \{x(n)\}.$$

They correspond to function space $\mathcal{M}_{\varphi,w}$ and sequence space $\mathfrak{m}_{\varphi,w}$, defined respectively as the set of $f \in L^0$ and $x = \{x(n)\}$ such that

$$\|f\|_{\mathcal{M}} = \inf\left\{\epsilon > 0 : P\left(\frac{f}{\epsilon}\right) \le 1\right\} < \infty \quad \text{and} \quad \|x\|_{\mathfrak{m}} = \inf\left\{\epsilon > 0 : p\left(\frac{x}{\epsilon}\right) \le 1\right\} < \infty.$$

Remark 4.2. By Corollary 2.11 it is clear that $P(f) \le M(f)$ and thus $M_{\varphi,w} \subset \mathcal{M}_{\varphi,w}$. Now, let $h \ge 0$ be a function such that $h \prec w$. Then $th^*(t) \le \int_0^t h^* = W(t)$, and so $h^*(t) \le w_1(t) := W(t)/t$. Note that w_1 is also a decreasing weight in I such that $w_1 \ge w$. Hence by Corollary 2.11,

$$\int_I \varphi\left(\frac{|f|}{h}\right) h \ge M_1(f) := \int_I \varphi\left(\frac{f^*}{w_1}\right) w_1 = \inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right) v : v^* \le w_1, v \ge 0\right\}.$$

Thus passing to infimum when $h \prec w$ we get $P(f) \ge M_1(f)$, and consequently we have

$$M_{\varphi,w} \subset \mathcal{M}_{\varphi,w} \subset M_{\varphi,w_1}. \tag{7}$$

Analogously $m_{\varphi,w} \subset \mathfrak{m}_{\varphi,w} \subset m_{\varphi,w_1}$. Moreover

$$\|f\|_M \ge \|f\|_{\mathcal{M}} \ge \|f\|_{M_1},$$

and in particular the functional $\|f\|_{\mathcal{M}}$ is faithful that is $\|f\|_{\mathcal{M}} = 0$ implies $f = 0$ as an element of L^0.

Proposition 4.3. *If the weight w is regular then the classes $\mathcal{M}_{\varphi,w}$ and $M_{\varphi,w}$ coincide and the associated functionals $\|\cdot\|_{\mathcal{M}}$ and $\|\cdot\|_M$ are equivalent.*

Proof. If the weight w is regular, the weights w and w_1 are equivalent. Hence $M_{\varphi,w} = M_{\varphi,w_1}$ and thus by (7), $\mathcal{M}_{\varphi,w} = M_{\varphi,w}$ (analogously $m_{\varphi,w} = \mathfrak{m}_{\varphi,w}$ when the weight w is regular). More precisely if $w_1 \le Cw$, $C \ge 1$, then $M_1(f) \ge CM(C^{-1}f) \ge M(C^{-1}f)$, and thus $\|f\|_{M_1} \ge C^{-1}\|f\|_M$. $\square$

A partial converse to Proposition 4.3 will be given later in Proposition 4.10.

We show now that $P(f)$ and $p(x)$ are rearrangement invariant.

Proposition 4.4. *For any function $f \in L^0$ and a sequence $x = \{x(n)\}$ we have $P(f) = P(f^*)$ and $p(x) = p(x^*)$.*

Proof. We provide the proof only in the case when $I = (0, \infty)$. Let $f \in L^0$.

First we show that $P(f^*) \ge P(f)$. Let $v \prec w$ and assume $\int_I \varphi\left(\frac{f^*}{v}\right) v < \infty$. It follows that $\lim_{t\to\infty} f^*(t) = 0$. Indeed, if we assume that $\inf_{t\ge 0} f^*(t) = K > 0$ then setting $B = \{t : v(t) \le b\}$, $b > 0$, we have $|B|\varphi\left(\frac{K}{b}\right) b \le \int_B \varphi\left(\frac{K}{v}\right) v \le \int_I \varphi\left(\frac{f^*}{v}\right) v < \infty$. Hence $|B| < \infty$ for any $b \ge 0$, and so $|\{t : v(t) > b\}| = \infty$ for every $b \ge 0$. Consequently $v^* = \infty$ on $(0, \infty)$, which contradicts the assumption $v \prec w$.

Thus by Ryff's theorem there exists an onto and measure preserving transformation $\tau : \operatorname{supp} f \to \operatorname{supp} f^*$ such that $|f| = f^* \circ \tau$ on $\operatorname{supp} f$. If $|\operatorname{supp} f| < \infty$ then $|\operatorname{supp} f| = |\operatorname{supp} f^*|$ and we extend τ to a measure preserving transformation from I onto I. If $|\operatorname{supp} f| = \infty$ then $\operatorname{supp} f^* = I$. In the first case define $v_1 = v \circ \tau$, in the second case let $v_1(t) = v \circ \tau(t)$ for $t \in \operatorname{supp} f$ and $v_1(t) = 0$ for $t \in (\operatorname{supp} f)^c$. Since the range of τ in both cases is $(0, \infty)$, so $v_1 \sim v$. Moreover in the first case,

$$\int_I \varphi\left(\frac{f^*}{v}\right) v = \int_I \varphi\left(\frac{f^* \circ \tau}{v \circ \tau}\right) v \circ \tau = \int_I \varphi\left(\frac{|f|}{v_1}\right) v_1,$$

and in the second case applying our convention we also get

$$\int_I \varphi\left(\frac{f^*}{v}\right) v = \int_{\operatorname{supp} f} \varphi\left(\frac{|f|}{v \circ \tau}\right) v \circ \tau = \int_I \varphi\left(\frac{|f|}{v_1}\right) v_1.$$

Summarizing, for every $v \prec w$ we found $v_1 \prec w$ such that $\int_I \varphi\left(\frac{f^*}{v}\right) v = \int_I \varphi\left(\frac{|f|}{v_1}\right) v_1$, which shows that $P(f^*) \geq P(f)$.

Now we will show that $P(f) \geq P(f^*)$. Let's consider first $f \in L^0$ such that its range is countable. Let $v \prec w$ and $\int_I \varphi\left(\frac{|f|}{v}\right) v < \infty$. Let $A = \{t : |f(t)| > K\}$ where $K = \lim_{t\to\infty} f^*(t)$.

Suppose $|A| = \infty$. Then since f has countable range there exists an onto and measure preserving transformation $\tau : (0,\infty) \to A$ such that $|f| \circ \tau = f^*$ on $(0,\infty)$. Therefore

$$\int_I \varphi\left(\frac{|f|}{v}\right) v \geq \int_A \varphi\left(\frac{|f|}{v}\right) v = \int_I \varphi\left(\frac{|f|\circ\tau}{v\circ\tau}\right) v\circ\tau = \int_I \varphi\left(\frac{f^*}{v\circ\tau}\right) v\circ\tau. \tag{8}$$

Since the range of τ is equal to A, setting $v_1 = v \circ \tau$ we have $v_1^* \leq v^*$, and hence $v_1 \prec w$.

Now let $|A| < \infty$. Setting $B = \{t : |f(t)| = K\}$, we have $|B| = \infty$. We find onto and measure preserving transformations $\tau_1 : (0,|A|) \to A$ and $\tau_2 : (|A|,\infty) \to B$ such that $|f| \circ \tau_1(t) = f^*(t)$ for $t \in (0,|A|)$, and $|f| \circ \tau_2(t) = f^*(t) = K$ for $t \in (|A|,\infty)$. Thus $\tau = \tau_1|_{(0,|A|)} + \tau_2|_{(|A|,\infty)}$ is a measure preserving mapping from $(0,\infty)$ onto $A \cup B$ and such that $|f| \circ \tau = f^*$ on I. Then

$$\int_I \varphi\left(\frac{|f|}{v}\right) v \geq \int_{A\cup B} \varphi\left(\frac{|f|}{v}\right) v = \int_I \varphi\left(\frac{|f|\circ\tau}{v\circ\tau}\right) v\circ\tau = \int_I \varphi\left(\frac{f^*}{v\circ\tau}\right) v\circ\tau. \tag{9}$$

We have that $(v \circ \tau)^* \leq v^*$ and so $v_1 = v \circ \tau \prec w$.

By (8) and (9), for any $v \prec w$ we can find $v_1 \prec w$ such that

$$\int_I \varphi\left(\frac{|f|}{v}\right) v \geq \int_I \varphi\left(\frac{f^*}{v_1}\right) v_1 \geq P(f^*). \tag{10}$$

Let now $f \in L^0$ and $v \prec w$ be such that $\int_I \varphi\left(\frac{|f|}{v}\right) v < \infty$. There exists a sequence $\{g_n\}$ of countable valued and measurable functions such that

$$|f| \leq g_n \leq v\varphi^{-1}\left(\left(1+\frac{1}{n}\right)\varphi\left(\frac{|f|}{v}\right)\right) \quad \text{and} \quad g_n \downarrow |f| \text{ a.e.}.$$

Hence $\varphi\left(\frac{g_n}{v}\right) v \downarrow \varphi\left(\frac{|f|}{v}\right) v$ and $\varphi\left(\frac{g_n}{v}\right) v \leq 2\varphi\left(\frac{|f|}{v}\right) v$ a.e., and by the Lebesgue convergence theorem,

$$\lim_{n\to\infty} \int_I \varphi\left(\frac{g_n}{v}\right) v = \int_I \varphi\left(\frac{|f|}{v}\right) v.$$

Hence for any $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$\int_I \varphi\left(\frac{|f|}{v}\right) v \geq \int_I \varphi\left(\frac{g_m}{v}\right) v - \epsilon.$$

Now by (10) there is $v_1 \prec w$ and such $\int_I \varphi\left(\frac{g_m}{v}\right) v \geq \int_I \varphi\left(\frac{g_m^*}{v_1}\right) v_1$. But $g_m^* \geq f^*$, and so

$$\int_I \varphi\left(\frac{|f|}{v}\right) v \geq \int_I \varphi\left(\frac{g_m^*}{v_1}\right) v_1 - \epsilon \geq \int_I \varphi\left(\frac{f^*}{v_1}\right) v_1 - \epsilon \geq P(f^*) - \epsilon.$$

□

The next corollary provides alternative formulas for $\|\cdot\|_P$ and $\|\cdot\|_p$. It follows by Proposition 4.4 and Theorem 2.5 in function case and Theorem 2.4 in sequence case.

Corollary 4.5. *For any $f \in L^0$ and $x = \{x(n)\}$,*

$$P(f) = \inf\left\{\int_I \varphi\left(\frac{f^*}{v}\right) v : v \prec w, v\downarrow\right\}, \quad p(x) = \inf\left\{\sum_{n=1}^\infty \varphi\left(\frac{x^*(n)}{v(n)}\right) v(n) : v \prec w, v\downarrow\right\},$$

where $v\downarrow$ denotes a non-negative decreasing function or sequence.

Lemma 4.6. *For any non-negative functions $f_n, f \in L^0$ such that $f_n \uparrow f$ and $\sup_n P(f_n) < \infty$, it holds $P(f_n) \uparrow P(f)$. The similar statement holds in sequence case.*

Proof. By Corollary 4.5, we can assume that each f_n is decreasing and for every $n \in \mathbb{N}$ there exists a non-negative decreasing v_n such that $v_n \prec w$ and $\frac{1}{n} + P(f_n) \geq \int_I \varphi\left(\frac{f_n}{v_n}\right) v_n$, $n \in \mathbb{N}$. By submajorization of v_n by w we have $v_n(t) \leq W(t)/t$, $n \in \mathbb{N}$. Hence the family $\{v_n\}$ is a sequence of decreasing functions bounded uniformly by a decreasing function on I. So, by Helly's selection theorem [17, Chapter 8, section 4], for some subsequence we have that $\lim_n v_n(t) = v(t)$ exists a.e. Obviously v is decreasing on I and $\int_0^t v \leq \underline{\lim}_n \int_0^t v_n \leq \int_0^t w$, and so $v \prec w$. Then by Fatou's Lemma we get

$$P(f) \geq \lim_n P(f_n) = \lim_n \int_I \varphi\left(\frac{f_n}{v_n}\right) v_n \geq \int_I \lim_n \varphi\left(\frac{f_n}{v_n}\right) v_n = \int_I \varphi\left(\frac{f}{v}\right) v \geq P(f),$$

and the proof is completed. □

Before the next proof, observe that the function $\psi(s,t) = \varphi(s/t)t$, $s,t > 0$, is convex with respect to two variables. In fact letting $s_i, t_i > 0$, $i = 1,2$, by convexity of φ, $\varphi\left(\frac{s_1+s_2}{t_1+t_2}\right) \leq \frac{t_1}{t_1+t_2}\varphi\left(\frac{s_1}{t_1}\right) + \frac{t_2}{t_1+t_2}\varphi\left(\frac{s_2}{t_2}\right)$. Hence $\psi\left(\frac{s_1+s_2}{2}, \frac{t_1+t_2}{2}\right) \leq \frac{1}{2}\varphi\left(\frac{s_1}{t_1}\right) t_1 + \frac{1}{2}\varphi\left(\frac{s_2}{t_2}\right) t_2 = \frac{1}{2}\psi(s_1,t_1) + \frac{1}{2}\psi(s_2,t_2)$.

Theorem 4.7. *The spaces $(\mathcal{M}_{\varphi,w}, \|\cdot\|_{\mathcal{M}})$ and $(\mathfrak{m}_{\varphi,w}, \|\cdot\|_{\mathfrak{m}})$ are rearrangement invariant Banach spaces satisfying the Fatou property.*

Proof. We prove it only in function case. We show first that $\|\cdot\|_{\mathcal{M}}$ is a norm. The fact that this functional is faithful was observed in Remark 4.2. It remains only to show that the functional P is convex for proving that its the homogeneous functional $\|\cdot\|_{\mathcal{M}}$ satisfies the triangle inequality. Let $P(f_i) < \infty$ for $i = 1,2$, and $\epsilon > 0$ be arbitrary. There exist h_i, $i = 1,2$, such that $h_i \prec w$ and

$$P(f_i) + \epsilon \geq \int_I \varphi\left(\frac{|f_i|}{h_i}\right) h_i, \qquad i = 1,2.$$

For $h = \frac{h_1+h_2}{2}$, by subadditivity $(h_1+h_2)^* \prec h_1^* + h_2^*$ we get $h \prec w$. Thus, in view of convexity of the function $(t,s) \mapsto \varphi(t/s)s$, $s,t > 0$, it follows

$$\begin{aligned} P\left(\frac{f_1+f_2}{2}\right) &\leq \int_I \varphi\left(\frac{|f_1+f_2|}{2h}\right) h \\ &\leq \frac{1}{2}\int_I \varphi\left(\frac{|f_1|}{h_1}\right) h_1 + \frac{1}{2}\int_I \varphi\left(\frac{|f_2|}{h_2}\right) h_2 \\ &\leq \frac{1}{2}\left(P(f_1) + P(f_2)\right) + \epsilon. \end{aligned}$$

Hence P is convex and so $\|\cdot\|_{\mathcal{M}}$ is a norm on $\mathcal{M}_{\varphi,w}$. It is rearrangement invariant in view of Proposition 4.4.

Finally let $f_n \in \mathcal{M}_{\varphi,w}$, $f \in L^0$ be non-negative, $f_n \uparrow f$ and $\sup_n \|f_n\|_{\mathcal{M}} = K < \infty$. Then $P\left(\frac{f_n}{K}\right) \leq 1$ for all $n \in \mathbb{N}$, and by Lemma 4.6, $P\left(\frac{f_n}{K}\right) \uparrow P\left(\frac{f}{K}\right) \leq 1$. Thus $\|f\|_{\mathcal{M}} \leq \sup_n \|f_n\|_{\mathcal{M}}$, and so $\|f\|_{\mathcal{M}} = \sup_n \|f_n\|_{\mathcal{M}}$. Therefore $\mathcal{M}_{\varphi,w}$ has the Fatou property, and thus it is complete in view of [1, Theorem 1.6]. □

Remark 4.8. If φ is equivalent to a p-concave Orlicz function for some $1 < p < \infty$ then $\mathcal{M}_{\varphi,w}$ is a p-concave Banach lattice. The proof is similar to that of Proposition 3.5 and Corollary 3.6.

4.2. **When do $\mathcal{M}_{\varphi,w}$ and $M_{\varphi,w}$ coincide?** We have seen in Proposition 4.3 that if w is regular then the classes $\mathcal{M}_{\varphi,w}$ and $M_{\varphi,w}$ coincide and the associated functionals $\|\cdot\|_{\mathcal{M}}$ and $\|\cdot\|_M$ are equivalent. In this subsection we shall give a partial converse in the case when $\alpha_\varphi > 1$. For this aim we shall compare the fundamental functions of $\mathcal{M}_{\varphi,w}$ and $M_{\varphi,w}$.

Proposition 4.9. *Let w be a weight function such that $W(t) < \infty$, $t \in I$. The fundamental function of the space $\mathcal{M}_{\varphi,w}$ is given by*

$$F_{\mathcal{M}}(t) = \|\chi_{(0,t)}\|_{\mathcal{M}} = \frac{t}{W(t)\varphi^{-1}(1/W(t))}, \quad t \in I.$$

The analogous formula is also valid in the sequence space $\mathfrak{m}_{\varphi,w}$.

Proof. For every constant $c > 0$ and every decreasing weight $v \prec w$ over $(0,t)$ we have by Jensen's inequality

$$\int_0^t \varphi\left(\frac{1}{cv(s)}\right)\frac{v(s)}{V(t)}\,ds \geq \varphi\left(\int_0^t \frac{1}{cV(t)}\,ds\right) = \varphi\left(\frac{t}{cV(t)}\right),$$

and since $V \leq W$ and for $\alpha > 0$ the function $t \mapsto t\varphi(\alpha/t)$ is decreasing for $t > 0$, we get

$$\int_0^t \varphi\left(\frac{1}{cv(s)}\right)v(s)\,ds \geq W(t)\varphi\left(\frac{t}{cW(t)}\right).$$

Taking the infimum with respect to decreasing weights $v \prec w$, by Corollary 4.5 we obtain

$$P\left(\frac{1}{c}\chi_{(0,t)}\right) \geq W(t)\varphi\left(\frac{t}{cW(t)}\right).$$

The weight $v_0 = \frac{W(t)}{t}\chi_{(0,t)}$ is decreasing with support $(0,t)$ and since $W(u)/u$ is decreasing we have for $0 \leq u \leq t$,

$$V_0(u) = \int_0^u v_0(s)\,ds = \frac{u}{t}W(t) \leq W(u),$$

while for $u > t$,

$$V_0(u) = V_0(t) = W(t) \leq W(u).$$

Hence $v_0 \prec w$, and so

$$P\left(\frac{1}{c}\chi_{(0,t)}\right) \leq \int_0^t \varphi\left(\frac{1}{cv_0(s)}\right)v_0(s)\,ds = \varphi\left(\frac{t}{c\,W(t)}\right)W(t).$$

Consequently for every $c > 0$ we obtain the equality

$$P\left(\frac{1}{c}\chi_{(0,t)}\right) = W(t)\varphi\left(\frac{t}{cW(t)}\right),$$

which implies that $F_{\mathcal{M}}(t) = \|\chi_{(0,t)}\|_{\mathcal{M}}$ is the unique solution c of $P\left(\frac{1}{c}\chi_{(0,t)}\right) = 1$, and gives the desired formula. $\square$

Proposition 4.10. *Let $\alpha_\varphi > 1$ and w be a weight function such that $W(t) < \infty$, $t \in I$. If the norm $\|\cdot\|_{\mathcal{M}}$ of $\mathcal{M}_{\varphi,w}$ and the functional $\|\cdot\|_M$ of $M_{\varphi,w}$ are equivalent on $\mathcal{M}_{\varphi,w}$ then the weight w is regular. In particular the spaces $\mathcal{M}_{\varphi,w}$ and $M_{\varphi,w}$ are equal with equivalent quasi-norms if and only if w is regular. A similar statement is valid for sequence spaces.*

Proof. We will conduct the proof only in function case when $I = (0,\infty)$. By hypothesis, for some constant C and every $t \in I$, $F_M(t) \leq CF_{\mathcal{M}}(t)$. Thus in view of inequalities (4) and Proposition 4.9,

$$\frac{1}{w(t)\varphi^{-1}\left(\frac{1}{tw(t)}\right)} \leq F_M(2t) \leq CF_{\mathcal{M}}(2t) = \frac{2Ct}{W(2t)\varphi^{-1}\left(\frac{1}{W(2t)}\right)}, \quad t > 0,$$

that is for $t > 0$,

$$W(2t)\varphi^{-1}\left(\frac{1}{W(2t)}\right) \le 2Ctw(t)\varphi^{-1}\left(\frac{1}{tw(t)}\right). \tag{11}$$

Since $\alpha_\varphi > 1$ so $\beta_{\varphi^{-1}} = 1/\alpha_\varphi < 1$, and hence for some $\varepsilon > 0$ and $K > 0$ we have for every $\lambda \ge 1$ and $u > 0$,

$$\varphi^{-1}(\lambda u) \le K\lambda^{1-\varepsilon}\varphi^{-1}(u).$$

Thus in view of $\frac{W(2t)}{tw(t)} > 1$ we have

$$2Ctw(t)\varphi^{-1}\left(\frac{1}{tw(t)}\right) < 2CKtw(t)\left(\frac{W(2t)}{tw(t)}\right)^{1-\varepsilon}\varphi^{-1}\left(\frac{1}{W(2t)}\right).$$

Combining the above with inequality (11) we get easily

$$\frac{tw(t)}{W(2t)} \ge \frac{1}{(2CK)^{1/\varepsilon}}, \quad t > 0.$$

Eventually, since $W(2t) \ge W(t)$ this implies that w is regular and the proof is completed. □

4.3. **Köthe duality.**

Proposition 4.11. *Assume that φ is N-function and $W(t) < \infty$ for $t \in I$. Then the Köthe dual of the space $\mathcal{M}_{\varphi,w}$ is the Orlicz-Lorentz space $\Lambda_{\varphi_*,w}$ equipped with its Orlicz norm, that is $\mathcal{M}'_{\varphi,w} = \Lambda^0_{\varphi_*,w}$ with equality of norms.*

Proof. Since $M_{\varphi,w} \subset \mathcal{M}_{\varphi,w}$, and the inclusion has norm not greater than one, we have

$$\mathcal{M}'_{\varphi,w} \subset M'_{\varphi,w} = \Lambda^0_{\varphi_*,w},$$

where the inclusion has norm not greater than one, and the equality is isometric by Theorem 3.10.

Conversely let us prove that $\Lambda^0_{\varphi_*,w} \subset \mathcal{M}'_{\varphi,w}$, and the inclusion has norm not greater than one. First we note that for any $g \in \Lambda^0_{\varphi_*,w}$ and any non-negative weight v with $v \prec w$ we have

$$\int_I \varphi_*(g^*)v \le \int_I \varphi_*(g^*)v^* \le \int_I \varphi_*(g^*)w,$$

since the function $\varphi_*(g^*)$ is decreasing. Thus $\Lambda^0_{\varphi_*,w} \subset \Lambda^0_{\varphi_*,v}$, and from the Amemiya formula for the Orlicz norm by Definition 3.9 we get

$$\|g\|^0_{\Lambda_{\varphi_*,v}} = \|g^*\|^0_{L_{\varphi_*}(v)} \le \|g^*\|^0_{L_{\varphi_*}(w)} = \|g\|^0_{\Lambda_{\varphi_*,w}}.$$

Fix now $g \in \Lambda^0_{\varphi_*,w}$. Then for any $v \prec w$ and $h \in M_{\varphi,v}$, we have

$$\int_I |hg| \le \|h\|_{M_{\varphi,v}}\|g\|^0_{\Lambda_{\varphi_*,v}} \le \|h\|_{M_{\varphi,v}}\|g\|^0_{\Lambda_{\varphi_*,w}}.$$

By Corollary 4.5, if $h \in \mathcal{M}_{\varphi,w}$ has norm $\|h\|_{\mathcal{M}_{\varphi,w}} < 1$ then there exists some decreasing v such that $v \prec w$ and $\int_I \varphi(h^*/v)v < 1$. Hence $\|h\|_{M_{\varphi,v}} \le 1$ and so

$$\int_I |hg| \le \|g\|_{\Lambda^0_{\varphi_*,w}}.$$

This shows that $g \in \mathcal{M}'_{\varphi,w}$ with norm $\|g\|_{\mathcal{M}'_{\varphi,w}} \le \|g\|_{\Lambda^0_{\varphi_*,w}}$. □

Corollary 4.12 (K. Leśnik)**.** *Assume that φ is N-function and $W(t) < \infty$ for $t \in I$. Then the space $\mathcal{M}_{\varphi,w}$ is equal to the Köthe dual of the Orlicz-Lorentz space $\Lambda^0_{\varphi_*,w}$, that is $(\Lambda^0_{\varphi_*,w})' = \mathcal{M}_{\varphi,w}$ with equality of norms.*

Proof. Since $\mathcal{M}_{\varphi,w}$ is a Köthe function space in the sense of [13, 1.b.17] and has the Fatou property, it holds by [13, p. 30, Remark 2] that it is equal to its Köthe bidual with equal norm. Then by Proposition 4.11

$$\mathcal{M}_{\varphi,w} = \mathcal{M}''_{\varphi,w} = (\Lambda^0_{\varphi_*,w})'.$$

□

Before we state the next result recall the definition of the Banach envelope of a quasi-Banach space X. Denote by X^* the dual space to X, that is the (Banach) space of bounded linear functionals. Let us define a functional on X by

$$|||x||| = \sup\{|f(x)| : f \in X^*, \text{ and } \|f\| \le 1\}.$$

If X^* separates the points of X then $|||\cdot|||$ is a norm on X. Then the *Banach envelope* $\widehat{X}$ of X is simply the completion of the normed linear space $(X, |||\cdot|||)$ [4, pp. 27-28].

Corollary 4.13. *Let φ be N-function and $W(t) < \infty$ for $t \in I$. The space $\mathcal{M}_{\varphi,w}$ is the Köthe bidual of $M_{\varphi,w}$. Consequently if $M_{\varphi,w}$ is a linear space and φ verifies condition Δ_2, then $\mathcal{M}_{\varphi,w}$ is the Banach envelope of $M_{\varphi,w}$.*

Proof. The first assertion is clear by Theorem 3.10 and Corollary 4.12. For the second one by Lemma 3.1 the assumption that $M_{\varphi,w}$ is linear implies that $\|\cdot\|_M$ is a quasi-norm. Now if φ satisfies condition Δ_2, then $M_{\varphi,w}$ is order continuous by the general results on the symmetrization of Banach function spaces (see [9, p. 279]). It results that the dual space $M^*_{\varphi,w}$ coincides with the Köthe dual $M'_{\varphi,w}$, and $|||\cdot|||$ is simply the norm induced on $M_{\varphi,w}$ by the norm of its Köthe bidual $\mathcal{M}_{\varphi,w}$. On the other hand, the condition Δ_2 for φ implies that $\beta_\varphi < \infty$ and so there exists $\beta_\varphi < p < \infty$ such that the function $\varphi(t^{1/p})/t$ is pseudo-decreasing, that is $\varphi(t^{1/p})/t \ge C\varphi(u^{1/p})/u$ for $0 < t < u$ and some $C > 0$. Hence $\varphi(t^{1/p})$ is equivalent to a concave function [16, 7], and by Remark 4.8, the Banach function space $\mathcal{M}_{\varphi,w}$ is p-concave. Then $\mathcal{M}_{\varphi,w}$ cannot contain an order isomorphic copy of ℓ_∞ and thus is order continuous (see e.g. [10]). Since $M_{\varphi,w}$ contains all simple integrable functions and $\mathcal{M}_{\varphi,w}$ is order continuous, then $\mathcal{M}_{\varphi,w}$ is the closure of all simple integrable functions [1], and thus we have $\widehat{M_{\varphi,w}} = \mathcal{M}_{\varphi,w}$. □

Proposition 4.14. *Assume that φ is N-function and that $W(t) < \infty$, $t \in I$. The following assertions are equivalent.*

(i) *$M_{\varphi,w}$ is a linear space and has a norm equivalent to the functional $\|\cdot\|_M$.*
(ii) *$M_{\varphi,w} = \mathcal{M}_{\varphi,w}$ and the functional $\|\cdot\|_M$ is equivalent to the norm $\|\cdot\|_{\mathcal{M}}$.*

If moreover $\alpha_\varphi > 1$ these conditions are equivalent to

(iii) *The weight w is regular.*

Proof. (ii) $\Longrightarrow$ (i) is clear since $\mathcal{M}_{\varphi,w}$ is a Banach function space.

(i) $\Longrightarrow$ (ii) Notice that the quasi-norm $\|\cdot\|_M$ has the Fatou property, by the general results on the symmetrization of quasi-Banach function spaces [9, p. 279]. Thus if $|||\cdot|||$ is an equivalent norm to $\|\cdot\|_M$ on $M_{\varphi,w}$, then $|||f|||_1 = \inf\{|||g||| : |f| \le g\}$ is a norm equivalent to $|||\cdot|||$ and so to $\|\cdot\|_1$, which preserves the order structure of $M_{\varphi,w}$ and satisfies the isomorphic Fatou property, that is if $f_n \in M_{\varphi,w}$, $f_n \uparrow f$ a.e., and $\sup_n |||f_n|||_1 < \infty$ then $f \in M_{\varphi,w}$ and $\sup_n |||f_n|||_1 \le C\,|||f|||_1$ for some $C > 0$ depending only on the norm $|||\cdot|||_1$. It is well known then that $M_{\varphi,w}$ can be renormed with an order compatible norm satisfying the usual (isometric) Fatou property, namely

$$|||f|||_2 = \inf\{\lim_n |||f_n|||_1 : 0 \le f_n \uparrow |f|\}$$

(see e.g. [19, pp. 446-452] where the isomorphic Fatou property is called the weak Fatou property). Then $(M_{\varphi,w}, |||\cdot|||_2)$ becomes a Banach function space with the Fatou property. Since $M_{\varphi,w}$ as well as its Köthe dual $\Lambda_{\varphi_*,w}$ contain the indicator functions of integrable sets, they become Köthe function spaces in the

sense of [13, 1.b.17]. Hence $(M_{\varphi,w}, |||\cdot|||_2)'' = (M_{\varphi,w}, |||\cdot|||_2)$ isometrically. By Corollary 4.13 we also have that $(M_{\varphi,w}, \|\cdot\|_M)'' = (\mathcal{M}_{\varphi,w}, \|\cdot\|_{\mathcal{M}})$. The equivalence of $|||\cdot|||_M$ and $\|\cdot\|_M$ propagates to their dual and bidual norms, so finally $M_{\varphi,w} = \mathcal{M}_{\varphi,w}$ as sets and the quasinorm $\|\cdot\|_M$ is equivalent to the norm $\|\cdot\|_{\mathcal{M}}$.

Finally the equivalence (ii) $\iff$ (iii) when $\alpha_\varphi > 1$ is simply Proposition 4.10. □

Example 4.15. Here is an example of a decreasing weight function w with $W(t) < \infty$, $t \in I$, and such that $1/w$ verifies condition Δ_2 but w is not regular. Consequently if $\alpha_\varphi > 1$ then the class $M_{\varphi,w}$ is linear and $\|\cdot\|_M$ is a quasi-norm, but it does not admit an equivalent norm.

Let $I = (0,1)$ and w be defined by $w(t) = \max(2^{-(k+1)^2} t^{-1}, 2^{k^2})$ when $t \in (4^{-(k+1)^2}, 4^{-k^2}]$, $k = 0, 1, \ldots$. Then $w(t) \le t^{-1/2}$ for all $t \in I$, and so $W(t) < \infty$ for $t \in I$. We also have that $w(st) \ge s^{-1} w(t)$ for all $s > 1$, $t > 0$ such that $st \le 1$, while $w(st_k) = s^{-1} w(t_k)$ for $t_k = 4^{-k^2}$, $k = 1, 2 \ldots$, and $1 \le s \le w(t_k)/w(t_{k-1}) = 2^{2k-1}$. This implies that $\alpha_w = -1$. Thus w is not regular by [6, Lemma 6], while $1/w$ verifies Δ_2 condition since $\beta_{1/w} = -\alpha_w = 1 < \infty$.

Remark 4.16. Let's recall the space which was discussed in [8]. For an Orlicz function φ and a weight function w let $S_{\varphi,w}$ be the space of $f \in L^0$ such that

$$\|f\|_S = \inf\{\epsilon > 0 : S(f/\epsilon) \le 1\} < \infty, \quad \text{where} \quad S(f) = \int_I \varphi\left(\frac{\int_0^t f^*}{W(t)}\right) w(t)\, dt.$$

Then $(S_{\varphi,w}, \|\cdot\|_S)$ is a r.i. Banach space (it is called $M_{\varphi,w}$ in [8]). It follows from [8, Theorem 3.1] that $S_{\varphi,w} = \Lambda'_{\varphi_* w}$ with equivalent norms whenever φ and its complementary function φ_*, satisfy condition Δ_2 and $W(\infty) = \infty$ (in the case where the interval I is infinite). However they are not equal neither to $M_{\varphi,w}$ nor $\mathcal{M}_{\varphi,w}$ without these assumptions.

Let's take for instance $w \equiv 1$ on I, and $\varphi(t) = t$, $t \ge 0$. Clearly φ_* does not satisfy Δ_2. Moreover $S_{\varphi,w} = LLogL$ if $a = 1$ and $S_{\varphi,w} = \{0\}$ in case when $a = \infty$ [1]. However $M_{\varphi,w} = \mathcal{M}_{\varphi,w} = L_1$ with equality of norms.

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